EXAMPLE 7.2

A real-valued random signal is observed in white noise which is uncorrelated with the signal. The observed sequence is given by

$$x[n] = s[n] + \eta[n]$$

where

$$R_s[l] = 2(0.8)^{|l|}\,; \qquad R_\eta[l] = 2\delta[l]$$

It is desired to estimate $s[n]$ using the present and previous two observations. This problem corresponds to the first line in the table of typical Wiener filtering problems, *i.e.,* the *filtering* of a signal in additive noise. Here $d[n] = s[n]$, so

$$R_d[l] = R_s[l] = 2(0.8)^{|l|}$$

$$\begin{aligned} R_{dx}[l] &= \mathcal{E}\left\{s[n]x[n-l]\right\} = \mathcal{E}\left\{s[n](s[n-l] + \eta[n-l])\right\} \\ &= R_s[l] = 2(0.8)^{|l|} \end{aligned}$$

and

$$\begin{aligned} R_x[l] &= \mathcal{E}\left\{(s[n] + \eta[n])(s[n-l] + \eta[n-l])\right\} \\ &= R_s[l] + R_\eta[l] = 2(0.8)^{|l|} + 2\delta[l] \end{aligned}$$

Then evaluating these expressions for the needed terms yields

$$R_d[0] = 2$$

$$R_x[0] = 4.00 \qquad R_x[1] = R_x[-1] = 1.60 \qquad R_x[2] = R_x[-2] = 1.28$$

$$R_{dx}[0] = 2.00 \qquad R_{dx}[1] = 1.60 \qquad R_{dx}[2] = 1.28$$

The Wiener-Hopf equation thus becomes

$$\begin{bmatrix} 4.00 & 1.60 & 1.28 \\ 1.60 & 4.00 & 1.60 \\ 1.28 & 1.60 & 4.00 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} 2.00 \\ 1.60 \\ 1.28 \end{bmatrix}$$

which can be solved to find

$$h[0] = 0.3824 \qquad h[1] = 0.2000 \qquad h[2] = 0.1176$$

The mean-square error is then given by

$$\begin{aligned} \sigma_\varepsilon^2 &= R_d[0] - \sum_{l=0}^{P-1} h^*[l] R_{dx}[l] \\ &= 2.000 - \sum_{l=0}^{2} h[l] \cdot 2(0.8)^{|l|} = 0.7647 \end{aligned}$$

$\square$

EXAMPLE 7.3

Let us consider the problem of estimating the sequence in Example 7.2 two points ahead, using the same three observations. This corresponds to the prediction problem listed as the second line in the table of typical Wiener filtering problems. Note that it is not the same as the problem of "linear prediction" which is represented in the fourth line of the table.

For this problem we have $d[n] = s[n+2]$ and

$$\begin{aligned} R_{dx}[l] &= \mathcal{E}\{s[n+2]x[n-l]\} = \mathcal{E}\{s[n+2](s[n-l]+\eta[n-l])\} \\ &= R_s[l+2] = 2(0.8)^{|l+2|} \end{aligned}$$

which yields the values

$$R_{dx}[0] = 1.280 \qquad R_{dx}[1] = 1.024 \qquad R_{dx}[2] = 0.8192$$

$R_d[l]$ and $R_x[l]$ are found to be the same as before.

The Wiener-Hopf equation becomes

$$\begin{bmatrix} 4.00 & 1.60 & 1.28 \\ 1.60 & 4.00 & 1.60 \\ 1.28 & 1.60 & 4.00 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} 1.280 \\ 1.024 \\ 0.8192 \end{bmatrix}$$

and solving for the filter terms yields

$$h[0] = 0.2447 \qquad h[1] = 0.1280 \qquad h[2] = 0.07529$$

The mean-square error in this case is

$$\sigma_\varepsilon^2 = 2.000 - \sum_{l=0}^{2} h[l] \cdot 2(0.8)^{|l+2|} = 1.494$$

This is larger than the mean-square error in the previous example by about a factor of 2. This indicates that prediction of the signal two points ahead is considerably less accurate than estimation of its current value. □