Expected Value of Random Variables

For a discrete random variable, *I*, the "expectation" or "expected value" of the random variable is the weighted sum:

$$E[I] = \sum_{i=-\infty}^{\infty} i \, \mathrm{f}_I[i]$$

For a continuous random variable, *X*, its expectation or expected value is a continuous weighted average :

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

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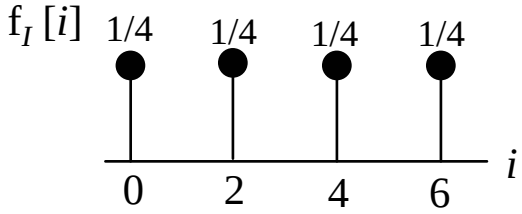
<u>Example</u>: Consider a discrete random variable I that takes values: 0, 2, 4, 6.

a) Let I be uniform; its expectation is same as the arithmetic average:

$$E[I] = \sum_{i=-\infty}^{\infty} i\, f_I[i] = \sum_{i=0}^{6} i\, f_I[i]$$

$$= \frac{0+2+4+6}{4} = 3.$$

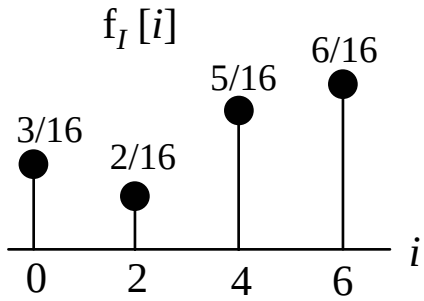


b) If the DPDF of I is as shown, its expected value is

$$E[I] = \sum_{i=-\infty}^{\infty} i\, f_I[i] = \sum_{i=0}^{6} i\, f_I[i]$$

$$= 0\times\frac{3}{16} + 2\times\frac{2}{16} + 4\times\frac{5}{16} + 6\times\frac{6}{16}$$

$$= \frac{0+4+20+36}{16} = \frac{15}{4} = 3.75.$$



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Invariance of Expectation

Let: $Y = g(X)$

We can show that:

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx = E[g(X)]$$

Where we have defined:

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx}$$

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Some Important Properties of Expectation

<u>Mean</u>: m_X

1. $E[c] = c$, c is a constant
2. $E[cX] = cE[X]$
3. $E[X + c] = E[X] + c$
4. Double expectation:

$$E[E[X]] = E[m_X] = m_X = E[X]$$

5. Expectation of a sum of two random variables:

$$E[X + Y] = E[X] + E[Y]$$

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"Moments" of Random Variables

n^{th} moment: $E\left[X^n\right] \quad n = 1, 2, 3, \ldots$

n^{th} *central* moment: $E\left[(X - m_X)^n\right] \quad n = 1, 2, 3, \ldots$

where $m_X = E\left[X\right]$ (the "mean")

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***n*th moment**

$$E\left[X^n\right] = \int_{-\infty}^{\infty} x^n \, f_X(x)\, dx \qquad E\left[I^n\right] = \sum_{i=-\infty}^{\infty} i^n \, \mathrm{f}_I[i]$$

For $n = 1$ (*mean* of the r.v.)

$$m_X = E[X] = \int_{-\infty}^{\infty} x \, f_X(x)\, dx \qquad m_I = E[I] = \sum_{i=-\infty}^{\infty} i \, \mathrm{f}_I[i]$$

For $n = 2$: second moment (average "power" of the r.v.)

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 \, f_X(x)\, dx \qquad E\left[I^2\right] = \sum_{i=-\infty}^{\infty} i^2 \, \mathrm{f}_I[i]$$

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***n*th central moment**

$$E\left[\left(X-m_X\right)^n\right]=\int_{-\infty}^{\infty}\left(x-m_X\right)^n f_X\left(x\right)dx \qquad E\left[\left(I-m_I\right)^n\right]=\sum_{i=-\infty}^{\infty}\left(i-m_I\right)^n \mathrm{f}_I\left[i\right]$$

For $n = 2$ (*variance* of the r.v.)

$$\sigma_X^2=E\left[\left(X-m_X\right)^2\right]=\int_{-\infty}^{\infty}\left(x-m_X\right)^2 f_X\left(x\right)dx$$

$$\sigma_I^2=E\left[\left(I-m_I\right)^2\right]=\sum_{i=-\infty}^{\infty}\left(i-m_I\right)^2 \mathrm{f}_I\left[i\right]$$

Standard deviation

$$\sigma_X=\sqrt{\sigma_X^2}$$

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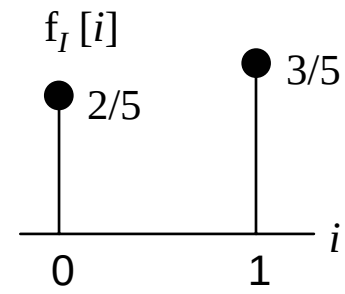
<u>Example</u>: Bernoulli r.v.



$$\Pr[1] = p = \frac{3}{5}, \quad \Pr[0] = 1 - p = \frac{2}{5}$$

$$m_I = E[I] = \sum_{i=0}^{1} i\, f_I[i] = \frac{2}{5}(0) + \frac{3}{5}(1) = \frac{3}{5}$$

$$\sigma_I^2 = E\left[(I - m_I)^2\right] = \sum_i (i - m_I)^2 f_I[i]$$

$$= \left(0 - \frac{3}{5}\right)^2 \frac{2}{5} + \left(1 - \frac{3}{5}\right)^2 \frac{3}{5}$$

$$= \frac{9}{25} \cdot \frac{2}{5} + \frac{4}{25} \cdot \frac{3}{5} = \frac{30}{125} = \frac{6}{25}$$

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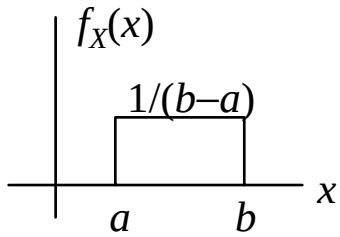
<u>Example</u>: X is uniform in the range $[a, b]$. Find m_X and σ_X^2.

$$\text{(a)}\quad m_X = E[X] = \frac{1}{b-a}\int_a^b x\,dx = \frac{b^2-a^2}{(b-a)2} = \frac{b+a}{2}$$



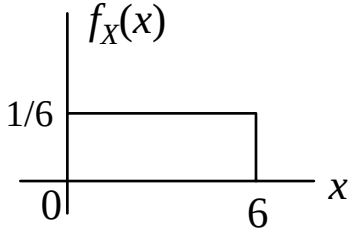
$$\text{(b)}\quad \sigma_X^2 = E\left[(X-m_X)^2\right] = \frac{1}{b-a}\int_a^b (x-m_X)^2\,dx$$

$$= \frac{1}{b-a}\cdot\frac{(x-m_X)^3}{3}\bigg|_a^b = \frac{(b-m_X)^3-(a-m_X)^3}{3(b-a)}$$

$$= \frac{\left(\frac{b-a}{2}\right)^3-\left(\frac{a-b}{2}\right)^3}{3(b-a)} = \frac{\frac{1}{4}(b-a)^3}{3(b-a)} = \frac{(b-a)^2}{12}$$

Let $a = 0$, $b = 6$, then

$$m_X = 3,\quad \sigma_X^2 = \frac{(6-0)^2}{12} = 3.$$



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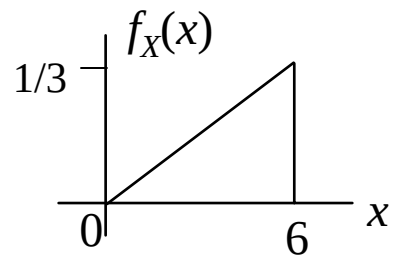
Example:

Now consider a random variable X with a probability density function:

$$f_X(x) = \begin{cases} \frac{1}{18}x & 0 \le x \le 6 \\ 0 & \text{otherwise.} \end{cases}$$



$$m_X = E[X] = \frac{1}{18}\int_0^6 x^2\,dx = \left.\frac{x^3}{54}\right|_0^6 = 4$$

$$\sigma_X^2 = E\left[(X - m_X)^2\right] = \frac{1}{18}\int_0^6 (x-4)^2\,x\,dx$$

$$= \frac{1}{18}\int_0^6 \left(x^3 - 8x^2 + 16x\right)dx = \frac{1}{18}\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^6 = 2.$$

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<u>More about Variance</u>: $\left(\text{var}[X] \text{ or } \sigma_X^2\right)$

1. $\boxed{\sigma_X^2 = E\left[X^2\right] - m_X^2}$

$$\begin{aligned}
\text{var}\left[X\right] &= E\left[\left(X - m_X\right)^2\right] \\
&= E\left[X^2 - 2m_X X + m_X^2\right] \\
&= E\left[X^2\right] - 2m_X E\left[X\right] + m_X^2 \\
&= E\left[X^2\right] - 2m_X \cdot m_X + m_X^2 \\
&= E\left[X^2\right] - m_X^2 = E\left[X^2\right] - \left\{E\left[X\right]\right\}^2
\end{aligned}$$

2. var $[c] = 0$

3. var $[X + c] =$ var $[X]$

4. var $[cX] = c^2$ var $[X]$

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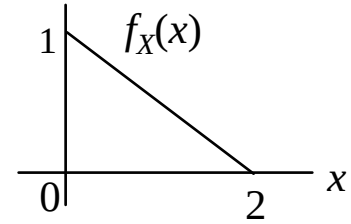
Example:

The pdf of a random variable X is given by



$$f_X(x) = \begin{cases} 1 - \dfrac{x}{2} & 0 \le X \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the mean, the second moment, and the variance of X.

$$m_X = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 x\left(1 - \frac{x}{2}\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^2 = \frac{2}{3}.$$

$$E\left[X^2\right] = \int_0^2 x^2\left(1 - \frac{x}{2}\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_0^2 = \frac{2}{3}.$$

$$\sigma_X^2 = \operatorname{var}[X] = \int_0^2 \left(x - \frac{2}{3}\right)^2 \left(1 - \frac{x}{2}\right) dx = \int_0^2 \left(-\frac{1}{2}x^3 + \frac{5}{3}x^2 - \frac{14}{9}x + \frac{4}{9}\right) dx$$

$$= \left[-\frac{1}{8}x^4 + \frac{5}{9}x^3 - \frac{7}{9}x^2 + \frac{4}{9}x\right]_0^2 = \frac{2}{9}.$$

Alternatively: $\sigma_X^2 = E\left[X^2\right] - \left\{E[X]\right\}^2 = \dfrac{2}{3} - \dfrac{4}{9} = \dfrac{2}{9}.$

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(b) Determine the mean and the variance of $2X + 3$.

$$E[2X+3] = E[2X] + E[3] = 2E[X] + 3 = 2\times\frac{2}{3} + 3 = \frac{13}{3}.$$

$$E\left[(2X+3)^2\right] = E\left[4X^2 + 12X + 9\right] = 4E\left[X^2\right] + 12E[X] + 9$$

$$= 4\times\frac{2}{3} + 12\times\frac{2}{3} + 9 = \frac{59}{3}.$$

$$\sigma^2_{2X+3} = E\left[(2X+3)^2\right] - \left\{E[2X+3]\right\}^2 = \frac{59}{3} - \left(\frac{13}{3}\right)^2 = \frac{8}{9}$$

$$or: \quad \sigma^2_{2X+3} = \mathrm{var}[2X+3] = \mathrm{var}[2X] = 4\,\mathrm{var}[X] = 4\times\frac{2}{9} = \frac{8}{9}.$$

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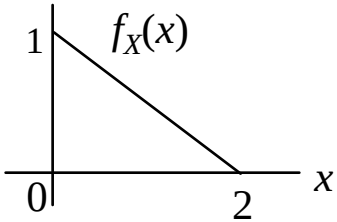
Example:

Now consider a random variable $Y = 2X + 3$, where the pdf of X is

$$f_X(x) = \begin{cases} 1 - \dfrac{x}{2} & 0 \le X \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

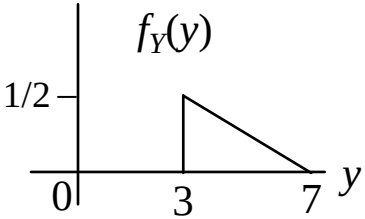


(a) Determine the pdf of Y.

Using the formula

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\Big|_{x=g^{-1}(y)}$$

we first obtain $\left|\dfrac{dy}{dx}\right| = 2$ and $x = \dfrac{y-3}{2}$,



and then we have

$$f_Y(y) = \frac{1}{2} f_X\left(\frac{y-3}{2}\right) = \frac{1}{8}(7-y), \qquad 3 \le y \le 7.$$

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(b) Determine the mean, second moment, and the variance of Y.

$$m_Y = E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \frac{1}{8}\int_3^7 y(7-y)\, dy = \left[\frac{7y^2}{16} - \frac{y^3}{24}\right]_3^7 = \frac{13}{3}.$$

$$E\left[Y^2\right] = \frac{1}{8}\int_3^7 y^2(7-y)\, dy = \left[\frac{7y^3}{24} - \frac{y^4}{32}\right]_3^7 = \frac{59}{3}.$$

$$\operatorname{var}[Y] = \sigma_Y^2 = \frac{1}{8}\int_3^7 \left(y - \frac{13}{3}\right)^2 (7-y)\, dy = \int_3^7 \left(-\frac{y^3}{8} + \frac{47}{24}y^2 - \frac{715}{72}y + \frac{1183}{72}\right) dy$$

$$= \left[-\frac{1}{32}y^4 + \frac{47}{72}y^3 - \frac{715}{144}y^2 + \frac{1183}{72}y\right]_3^7 = \frac{8}{9}.$$

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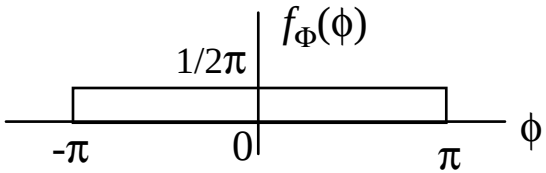
Example:

$X = A\cos(\omega t_0 + \phi)$ ϕ is a uniform random variable: $[-\pi, \pi]$

$X = g(\phi)$

$E[X] = E_\phi[g(\phi)] = \int g(\phi) f_\Phi(\phi)\, d\phi$



(a) Mean

$$
\begin{aligned}
m_X &= E\left[A\cos\left(\omega t_0 + \phi\right)\right] \\
&= A\int_{-\pi}^{\pi} \cos\left(\omega t_0 + \phi\right)\frac{1}{2\pi}\, d\phi \\
&= \frac{A}{2\pi}\cdot \sin\left(\omega t_0 + \phi\right)\Big|_{-\pi}^{\pi} \\
&= \frac{A}{2\pi}\left[\sin\left(\omega t_0 + \pi\right) - \sin\left(\omega t_0 - \pi\right)\right] \\
&= \frac{A}{2\pi}\left[\sin \omega t_0 \cos\pi + \cos\omega t_0 \underbrace{\sin\pi}_{0} - \sin\omega t_0 \cos\pi + \cos\omega t_0 \underbrace{\sin\pi}_{0}\right] = 0
\end{aligned}
$$

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(b) Variance

$$\sigma_X^2 = E\left[A^2\cos^2\left(\omega t_0+\phi\right)\right]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos^2\left(\omega t_0+\phi\right)d\phi = \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\frac{1+\cos\left(2\omega t_0+2\phi\right)}{2}d\phi$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}d\phi + \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\cos\left(2\omega t_0+2\phi\right)d\phi$$

$$= \frac{A^2}{4\pi}\cdot\phi\Big|_{-\pi}^{\pi} + \frac{A^2}{4\pi}\frac{\sin\left(2\omega t_0+2\phi\right)}{2}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin\left(2\omega t_0+2\pi\right)-\sin\left(2\omega t_0-2\pi\right)\right]$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin 2\omega t_0 - \sin 2\omega t_0\right] = \frac{A^2}{2}$$

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<u>Example</u>: Exponential r.v. $f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$

(a) Mean

$$m_X = \int_0^\infty x\,\lambda e^{-\lambda x} dx \qquad \text{(integration by parts)}$$

$$= x \frac{\lambda e^{-\lambda x}}{-\lambda}\bigg|_0^\infty - \int_0^\infty \frac{\lambda e^{-\lambda x}}{-\lambda} dx$$

$$= \lim_{x\to\infty}\left[-x \cdot e^{-\lambda x}\right] + \frac{e^{-\lambda x}}{-\lambda}\bigg|_0^\infty = 0 + \frac{1}{\lambda}$$

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(b) Variance

First, find the second moment

$$E\left[X^2\right] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$= x^2 \frac{\lambda e^{-\lambda x}}{-\lambda}\Bigg|_0^\infty + \int_0^\infty \frac{2x\lambda e^{-\lambda x}}{\lambda} dx$$

$$= 0 + \frac{2xe^{-\lambda x}}{-\lambda}\Bigg|_0^\infty + \int_0^\infty \frac{2e^{-\lambda x}}{\lambda} dx$$

$$= 0 + 0 + 2\frac{e^{-\lambda x}}{-\lambda^2}\Bigg|_0^\infty = \frac{2}{\lambda^2}$$

The variance is then given by

$$\sigma_X^2 = E\left[X^2\right] - m_X^2 = \frac{2}{\lambda^2} - \left[\frac{1}{\lambda}\right]^2 = \frac{1}{\lambda^2}$$

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<u>Example</u>:

For a geometric random variable I with values in the range [0, ∞), determine the mean and the variance:

$$f_I[i] = p(1-p)^i, \quad i = 0,1,2,\cdots,\infty$$

Mean

$$E[I] = \sum_{i=0}^{\infty} i\, f_I[i] = p\sum_{i=0}^{\infty} i(1-p)^i = p\sum_{i=0}^{\infty}(1-p)\left\{i(1-p)^{i-1}\right\}$$

Note that

$$i(1-p)^{i-1} = -\frac{d}{dp}\left[(1-p)^i\right]$$

therefore we have

$$E[I] = p(1-p)\sum_{i=0}^{\infty}\left\{-\frac{d}{dp}\left[(1-p)^i\right]\right\} = -p(1-p)\frac{d}{dp}\sum_{i=0}^{\infty}(1-p)^i$$

$$E[I] = -p(1-p)\frac{d}{dp}\left(\frac{1}{1-(1-p)}\right) = -p(1-p)\left[-\frac{1}{p^2}\right] = \frac{1-p}{p}.$$

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Second Moment

$$E\left[I^2\right] = \sum_{i=0}^{\infty} i^2 \; f_I[i] = p\sum_{i=0}^{\infty} i^2 (1-p)^i = p\sum_{i=0}^{\infty} i(1-p)\left\{i(1-p)^{i-1}\right\}$$

$$= p(1-p)\sum_{i=0}^{\infty} i\left\{-\frac{d}{dp}\left[(1-p)^i\right]\right\} = -p(1-p)\frac{d}{dp}\underbrace{\sum_{i=0}^{\infty} i(1-p)^i}_{\text{mean}/p}.$$

$$E\left[I^2\right] = p(1-p)\frac{d}{dp}\left(\frac{1-p}{p^2}\right) = -p(1-p)\left(-\frac{2-p}{p^3}\right) = \frac{p^2-3p+2}{p^2}$$

Variance

$$\sigma_I^2 = E\left[I^2\right] - \left\{E[I]\right\}^2 = \frac{p^2-3p+2}{p^2} - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}$$

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Transform Methods

The Moment-Generating Function (MGF)

$$M_X(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} f_X(x) e^{sx} dx$$

$\Rightarrow$ Laplace transform of pdf (with a sign change!)

<u>Example</u>:

Find the MGF of an exponential r.v.

$$M_X(s) = \int_0^{\infty} e^{-\lambda x} e^{sx} dx = \lambda \int_0^{\infty} e^{-(\lambda - s)x} dx$$

$$= \frac{\lambda}{-(\lambda - s)} e^{-(\lambda - s)x} \Big|_0^{\infty} = \frac{\lambda}{\lambda - s} \qquad (\mathrm{Re}[s] < \lambda)$$

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The nth moment (moment theorem)

$$E\left[X^n\right] = \left.\frac{d^n M_X(s)}{ds^n}\right|_{s=0}$$

<u>Example</u>: (exponential r.v.)

$$M_X(s) = \frac{\lambda}{\lambda - s}$$

(a) Let $n = 1$ $\quad E[X] = \left.\frac{dM_X(s)}{ds}\right|_{s=0} = -\left.\frac{\lambda}{(\lambda - s)^2}(-1)\right|_{s=0} = \frac{1}{\lambda}$

(b) Let $n = 2$ $\quad E\left[X^2\right] = +\left.\frac{2\lambda}{(\lambda - s)^3}\right|_{s=0} = \frac{2}{\lambda^2}$

(c) Variance

$$\sigma^2 = E\left[X^2\right] - \left(E[X]\right)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

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The Probability Generating Function (discrete r.v.)

$$G_I(z) = E\left[z^i\right] = \sum_{i=-\infty}^{\infty} \mathrm{f}_I[i]\, z^i$$

$\Rightarrow$ z-transform of PMF (with a sign change!)

<u>Example</u>: Poisson r.v.

$$G_I(z) = \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} e^{-\alpha} z^i = e^{-\alpha} \sum_{i=0}^{\infty} \frac{(\alpha z)^i}{i!} = e^{\alpha(z-1)}$$

Generates Probabilities if I takes on non-negative integer values:

$$\frac{1}{n!} \frac{d^n G_I(z)}{dz^n}\bigg|_{z=0} = \mathrm{f}_I[n] = \Pr[I = n]$$

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Moments

$$E\left[I\right] = \left.\frac{dG_I\left(z\right)}{dz}\right|_{z=1}$$

$$E\left[I^2\right] = \left.\frac{d^2G_I\left(z\right)}{dz^2}\right|_{z=1} + \left.\frac{dG_I\left(z\right)}{dz}\right|_{z=1}$$

<u>Example</u>: (Poisson r.v.) $G_I(z) = e^{\alpha(z-1)}$

$$E\left[I\right] = \left.\frac{dG_I\left(z\right)}{dz}\right|_{z=1} = e^{-\alpha}\alpha e^{\alpha z}\Big|_{z=1} = \alpha$$

$$\left.\frac{d^2G_I\left(z\right)}{dz^2}\right|_{z=1} = e^{-\alpha}\alpha^2 e^{\alpha z}\Big|_{z=1} = \alpha^2$$

$$\therefore \quad E\left[I^2\right] = \alpha^2 + \alpha$$

$$\sigma_I^2 = E\left[I^2\right] - \left(E\left[I\right]\right)^2 = \alpha^2 + \alpha - \left(\alpha\right)^2 = \alpha$$

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<u>Example</u>: Geometric r.v.

Probability generating function

$$G_I(z) = \sum_{i=0}^{\infty} p(1-p)^i z^i = p\sum_{i=0}^{\infty}\left((1-p)z\right)^i = p \cdot \frac{1}{1-(1-p)z}$$

Mean: $$E[I] = \left.\frac{dG_I(z)}{dz}\right|_{z=1} = \left.\frac{p(1-p)}{\left(1-(1-p)z\right)^2}\right|_{z=1} = \frac{1-p}{p}$$

Variance:

$$E\left[I^2\right] = \left.\frac{d^2G_I(z)}{dz^2}\right|_{z=1} + \left.\frac{dG_I(z)}{dz}\right|_{z=1} = \left.\frac{d}{dz}\left[\frac{p(1-p)}{\left(1-(1-p)z\right)^2}\right]\right|_{z=1} + \frac{1-p}{p}$$

$$= \left.\frac{2p(1-p)^2}{\left(1-(1-p)z\right)^3}\right|_{z=1} + \frac{1-p}{p} = 2\left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p}$$

$$\sigma_I^2 = E\left[I^2\right] - \left(E[I]\right)^2 = 2\left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p} - \left(\frac{1-p}{p}\right)^2 = \left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p} = \frac{1-p}{p^2}$$

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Summary and Relations

MGF

$$M(s)$$

$$M(j\omega)$$ is called the <u>Characteristic Function</u>

PGF

$$G(z)$$

For a discrete random variable: $M(s) = G(z)\Big|_{z=e^s}$

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<u>Example</u>: Shannon-Fano coding

Symbol probabilities and Shannon-Fano codewords for text message "ELECTRICAL ENGINEERING": (space character ignored)

Letter	E	I	N	L	C	R	G	T	A
Probability	5/21	3/21	3/21	2/21	2/21	2/21	2/21	1/21	1/21
Codeword	00	010	011	100	1010	1011	110	1110	1111
Length, L	2	3	3	3	4	4	3	4	4

The average codeword length and its variance:

$$m_L = E[L] = \frac{5}{21}\times 2 + \frac{3}{21}\times 3 + \frac{3}{21}\times 3 + \frac{2}{21}\times 3 + \frac{2}{21}\times 4$$
$$+ \frac{2}{21}\times 4 + \frac{2}{21}\times 3 + \frac{1}{21}\times 4 + \frac{1}{21}\times 4 = 3.0476.$$

$$\sigma_I^2 = E\left[\left(L - m_L\right)^2\right] = \left(2 - m_L\right)^2 \frac{5}{21} + \left(3 - m_L\right)^2 \frac{3}{21} + \left(3 - m_L\right)^2 \frac{3}{21}$$
$$+ \left(3 - m_L\right)^2 \frac{2}{21} + \left(4 - m_L\right)^2 \frac{2}{21} + \left(4 - m_L\right)^2 \frac{2}{21} + \left(3 - m_L\right)^2 \frac{2}{21}$$
$$+ \left(4 - m_L\right)^2 \frac{1}{21} + \left(4 - m_L\right)^2 \frac{1}{21} = 0.5215.$$

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<u>Example</u>: Shannon-Fano coding continued

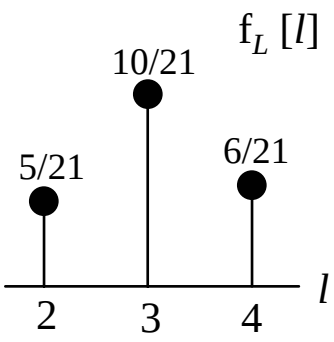
The random variable is codeword length, L, which takes values: 2, 3, 4.

$$m_L = E[L] = \sum_{l=2}^{4} l\, \mathrm{f}_L[l] = \frac{5}{21}\times 2 + \frac{10}{21}\times 3 + \frac{6}{21}\times 4 = \frac{64}{21} = 3.0476$$

$$\sigma_L^2 = E\left[\left(L - m_L\right)^2\right] = \sum_{l=2}^{4}\left(l - m_L\right)^2 \mathrm{f}_L(l)$$

$$= \left(2 - \frac{64}{21}\right)^2 \frac{5}{21} + \left(3 - \frac{64}{21}\right)^2 \frac{10}{21} + \left(4 - \frac{64}{21}\right)^2 \frac{6}{21}$$

$$= \frac{4830}{9261} = 0.5215.$$



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<u>Example</u>: Entropy

Entropy is defined as the *average* information associated with a set of symbols:

$$H = \sum_i \Pr[A_i] i[A_i] = -\sum_i \Pr[A_i] \log_x \Pr[A_i].$$

where *Ai* represents the symbols.

Consider the symbol set used in the text message "ELECTRICAL ENGINEERING" (ignore the space character) and their probabilities:

Letter	E	I	N	L	C	R	G	T	A
Probability	5/21	3/21	3/21	2/21	2/21	2/21	2/21	1/21	1/21

The entropy of this sequence (or optimal codeword length) is

$$L_H = -\frac{5}{21}\log_2\left(\frac{5}{21}\right) - \frac{3\times 3}{21}\log_2\left(\frac{3}{21}\right) - \frac{4\times 2}{21}\log_2\left(\frac{2}{21}\right) - \frac{2\times 1}{21}\log_2\left(\frac{1}{21}\right) = 3.0057.$$

Compare this value to m_L = 3.0476 in the previous example.

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