O-1 Fourier Analysis Review

Fourier Series (FS)
- definition
- coefficient computation
- useful properties
- Gibbs' phenomenon
- FS convergence
- FS for discrete time periodic signals

Fourier Transform
- Dirichlet's conditions
- Fourier transform pairs & properties

Discrete time Fourier Transform (DTFT)
- definition
- sampling issues, Nyquist rate

Discrete Fourier Transform
- definition
- MATLAB and DFT

Short Time Fourier Transform (STFT)

O-1 Fourier Analysis

I. Fourier Series

1) Basic idea

- to be able to represent a signal as a linear combination of basic signals
- to take advantage of LTI system properties

2) Fourier series (for a periodic signal)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk2\pi f_0 t}$$

$$k = 0 \rightarrow$$

$$k = \pm 1 \rightarrow$$

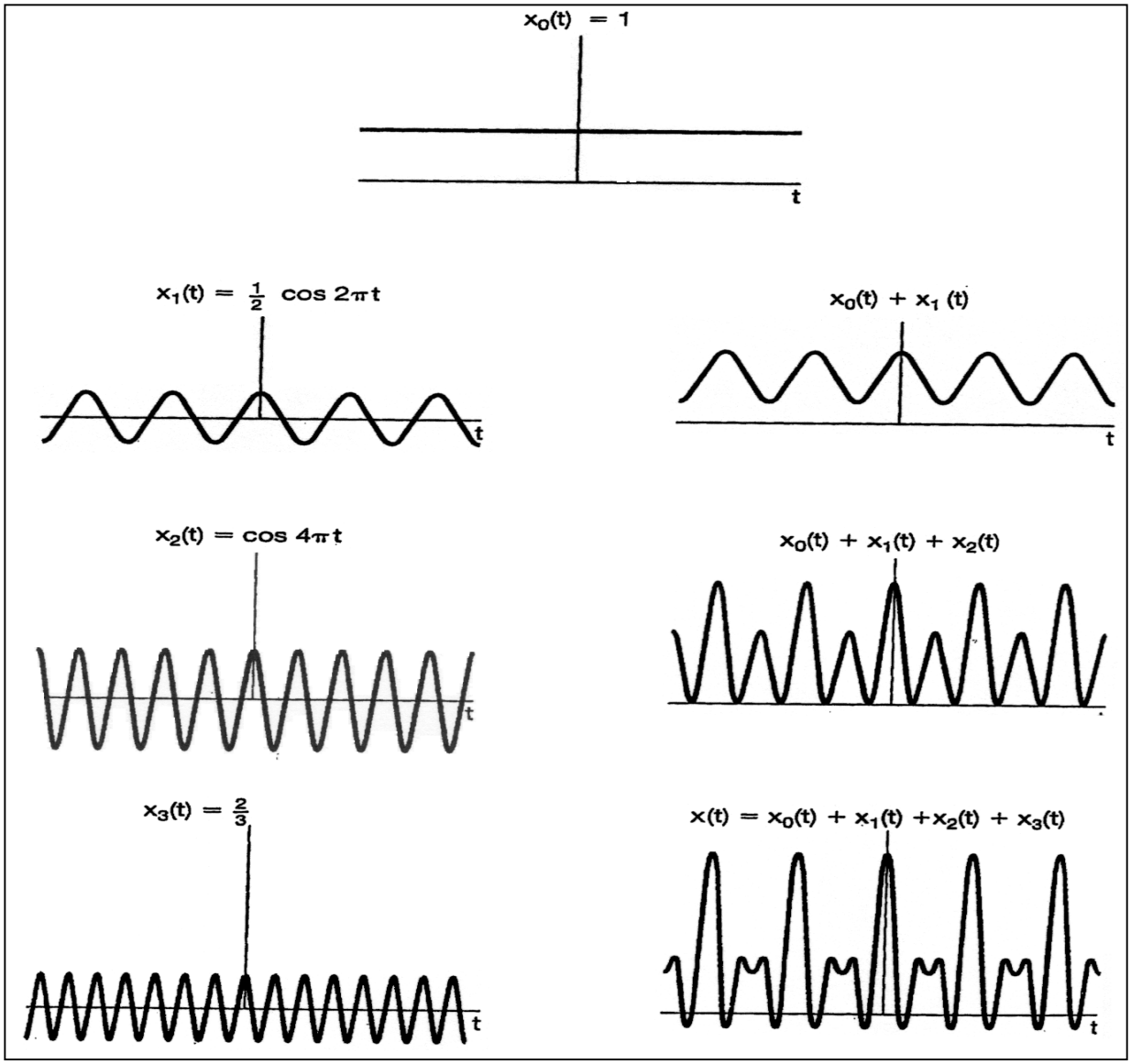
$$k = \pm 2 \rightarrow$$

- Fourier series property for a real, periodic $x(t)$

$$x(t) = x^*(t)$$

- alternate Fourier series representation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk2\pi f_0 t}$$


$x_0(t) = 1$
t
$x_1(t) = \frac{1}{2} \cos 2\pi t$
t
$x_0(t) + x_1(t)$
t
$x_2(t) = \cos 4\pi t$
t
$x_0(t) + x_1(t) + x_2(t)$
$x_3(t) = \frac{2}{3}$
t
$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t)$
t

3) How to define a_k ?

$$x(t) = \sum_{-\infty}^{+\infty} a_k e^{jk2\pi f_0 t}$$

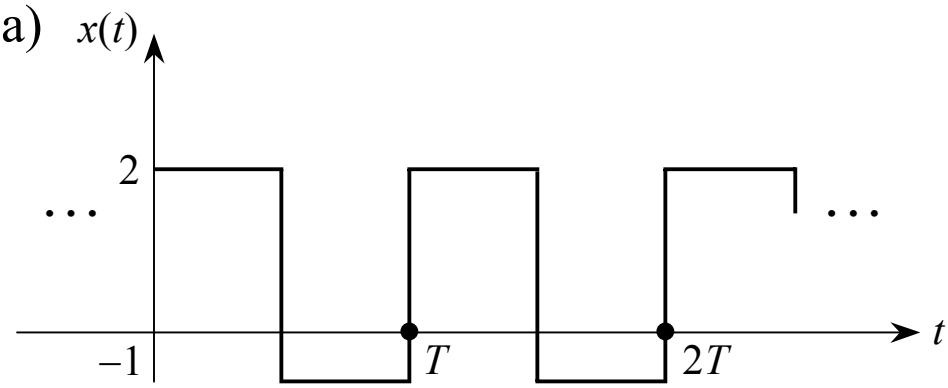
(1) $x(t) \times e^{-j2\pi n f_0 t} =$

(2) Integrate over 1 period to

$$\int_0^{T_0} x(t) e^{-jn2\pi n f_0 t} dt =$$

4) Example

a)



b) $x(t) = 3\cos 2t - 5\cos \pi t$

5) Useful Fourier series coefficient properties

$$x(t) \quad \text{real} + \text{even} \quad \Rightarrow \quad a_k$$

$$x(t) \quad \text{real} + \text{odd} \quad \Rightarrow \quad a_k$$

$x(t)$ and $y(t)$ periodic with same period T

Property	Signal	Coefficients
linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
time shift	$x(t - t_0)$	$a_k e^{-jk2\pi f_0 t_0}$
time reversal	$x(-t)$	a_{-k}
conjugation	$x^*(t)$	a_{-k}^*

Parseval's relation:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |a_k|^2$$

$$x(t) \leftrightarrow a_k$$

$$y(t) \leftrightarrow b_k$$

6) Truncation Fourier series – Gibbs' phenomenon

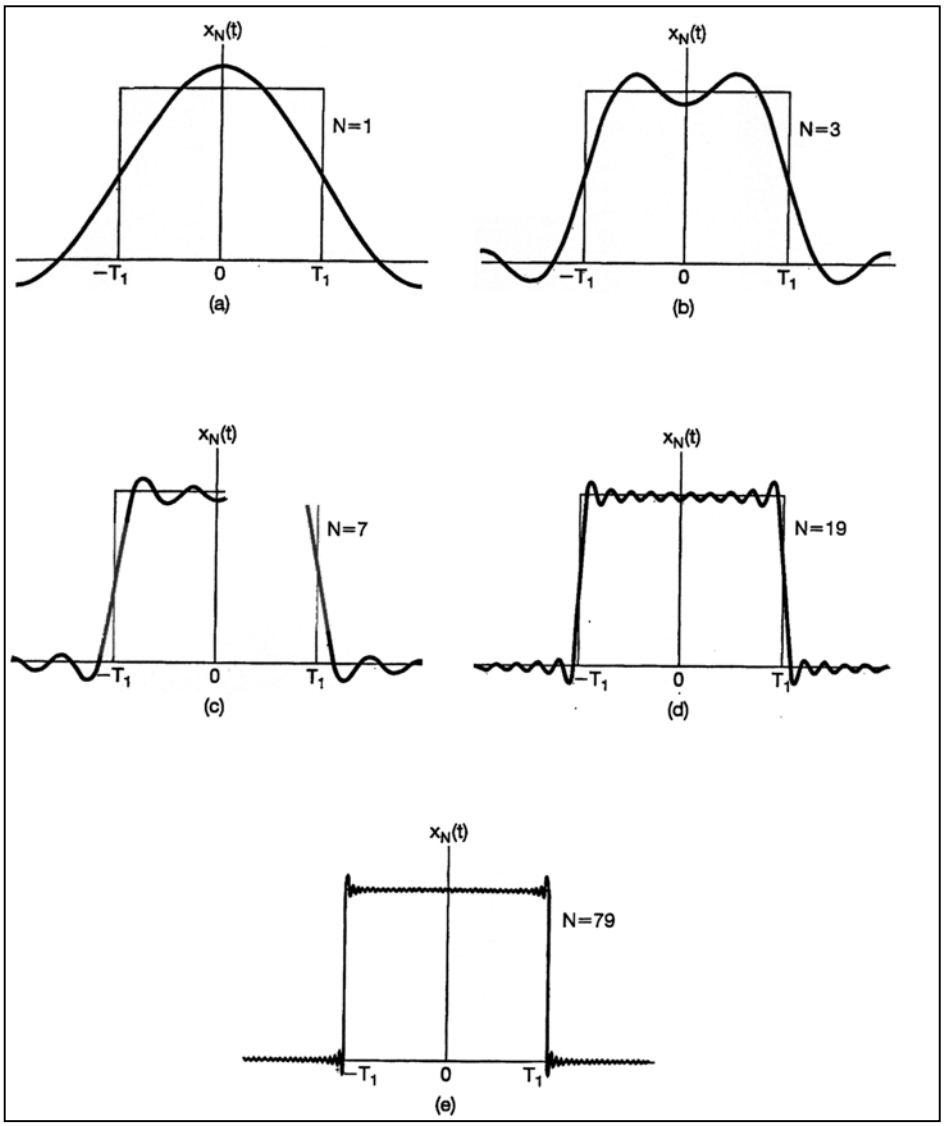


Figure. Convergence of the Fourier series representation as a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation

$$x_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t}$$ for several values of *N*.

7) Fourier series convergence

- not all periodic functions can be represented by Fourier series

  ↳ need to satisfy Dirichlet's conditions

(1) $x(t)$ absolutely integrable over a period T_0

(2) $x(t)$ has bounded variations: no more than a finite number of maxima or minima over period T_0

(3) $x(t)$ has a finite number of discontinuities

Table. Properties of Continuous-Time Fourier Series

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		Differentiation	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Differentiat		Integration	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Differentiation	Diffe	Different	$a_k = a^*_{-k}$ $\mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\}$ $\mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\}$ $\lvert a_k\rvert = \lvert a_{-k}\rvert$ $\sphericalangle a_k = -\sphericalangle a_{-k}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

8) Fourier series for a discrete-time periodic signal

- main difference with results obtained for continuous periodic signals:

  ↳ signal expressed as a <u>finite</u> series

$$x[n] = \sum_{\langle N \rangle} a_k \, e^{j2\pi kn/N}; \qquad x[n] = x[n+N]$$

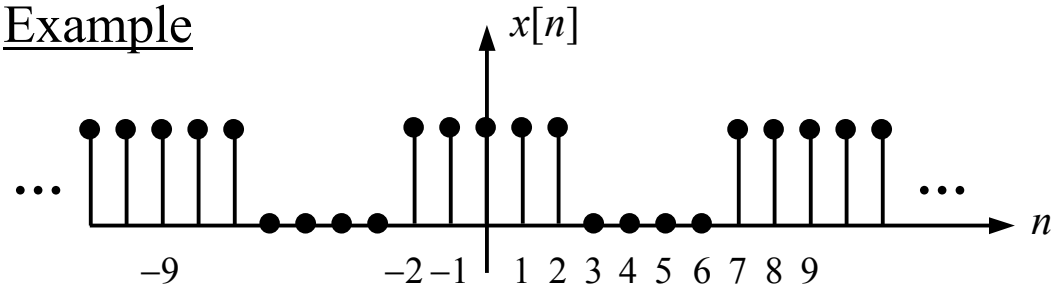
$$a_k =$$

- Partial Synthesis Issues

$$x[n] = \sum_{\langle N \rangle} a_k \, e^{jk2\pi n/N} \qquad N\text{: period of } x[n]$$

$$= \sum$$

<u>Example</u>



M=1

M=2

M=3

M=4

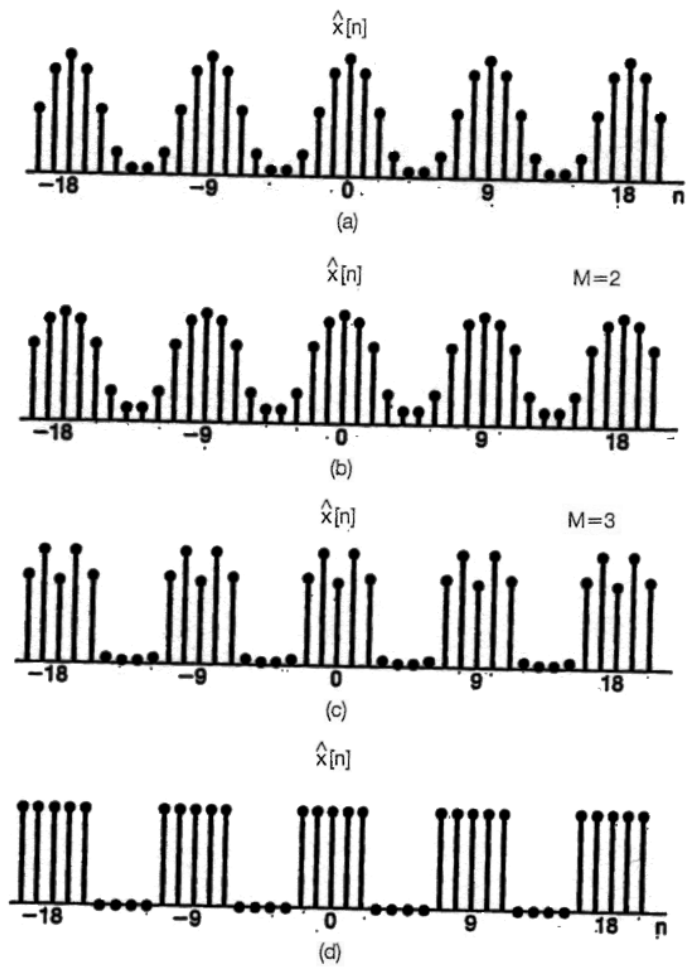


Table. Properties of Discrete-Time Fourier Series

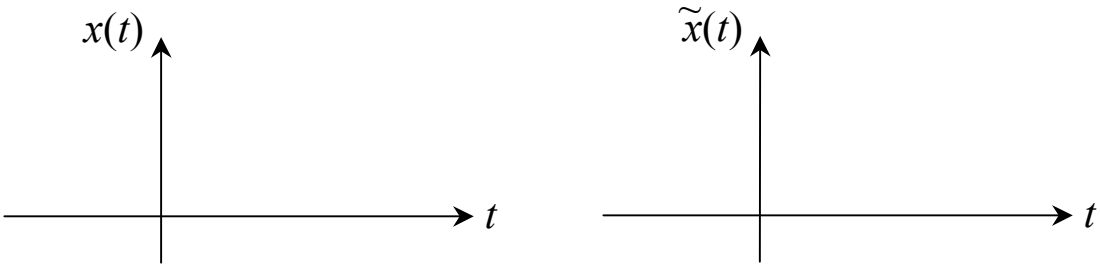
Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

II. Fourier Transform

- **Basic idea:** to represent a non-periodic signal



$$X(f) =$$

$$x(t) =$$

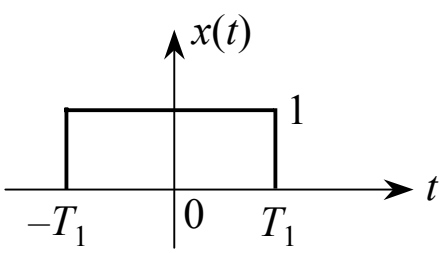
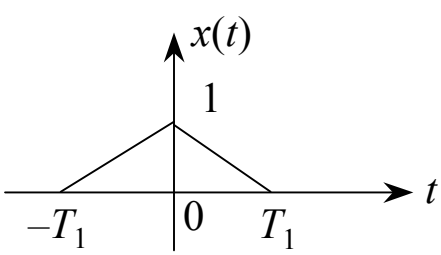
- **Fourier transform: Dirichlet's conditions**

(1) $x(t)$ absolutely integrable

(2) $x(t)$ has a finite number of maxima and minima within any finite interval

(3) $x(t)$ has a finite number of discontinuities within a finite interval

1) Basic Fourier transform pairs

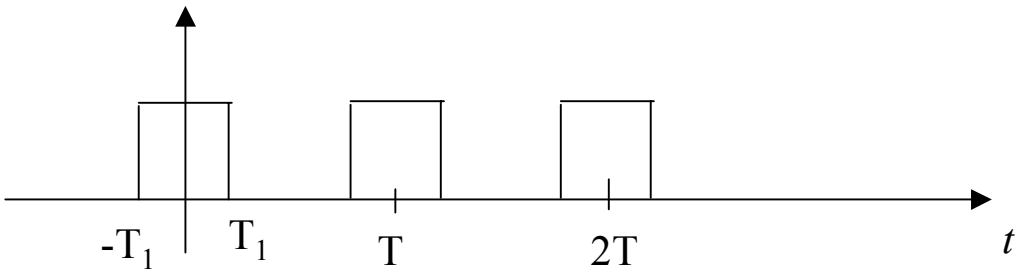
$x(t)$	$X(f)$
$\delta(t)$	
1	
sqn (t)	
$u(t)$	
	
	$T_1\ Sa^2\left(\pi f T_1\right) = T_1\ \mathrm{sinc}^2\left(f T_1\right)$
$\sum \delta(t-kT)$	$\frac{1}{T}\sum \delta(f-k/T)$

2) Fourier transform for a periodic signal

$$F_T\left\{1\right\} =$$

$$F_T\left\{e^{j2\pi f_0 t}\right\} =$$

$$F_T\left\{\sum_k a_k\, e^{j2\pi k f_0 t}\right\} =$$


-T1
T1
T
2T
t

3) Basic Fourier transform properties

linearity	$ax_1(t)+bx_2(t)$	$a_1X_1(f)+bX_2(f)$
time-shift	$x(t-t_0)$	$X(f)e^{-j2\pi ft_0}$
scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{\lvert a\rvert}\right)$
convolution	$x_1(t)*x_2(t)$	$X_1(f)\cdot X_2(f)$
differentiation	$\frac{d}{dt}x(t)$	$2\pi jf\,X(f)$
integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{X(f)}{2\pi jf}+\frac{1}{2}X(0)\delta(f)$
duality	$X(f)$	$x(-f)$
modulation	$x(t)e^{j2\pi ft_0}$	$X(f-f_0)$
Parseval's relation	$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2\,dt$	$=\int_{-\infty}^{+\infty}\lvert X(f)\rvert^2\,df$
modulation	$x_1(t)\cdot x_2(t)$	$X_1(f)*X_2(f)$

- Fourier transform properties

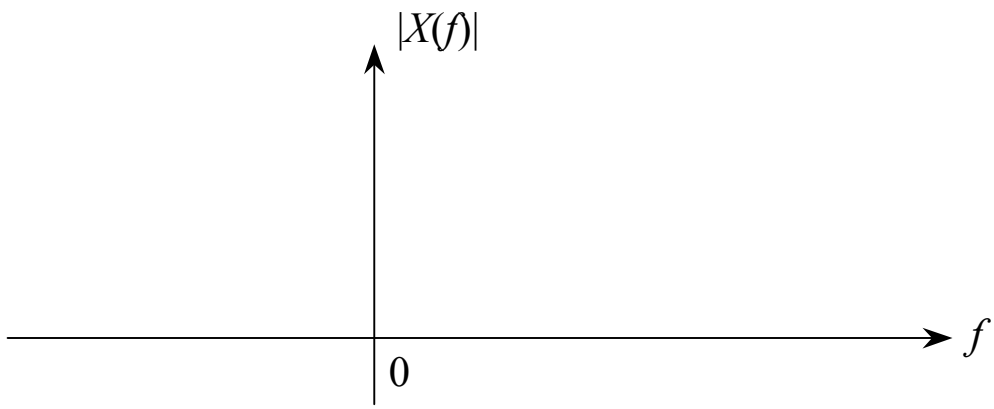
$x(t)$ real signal $\Rightarrow$

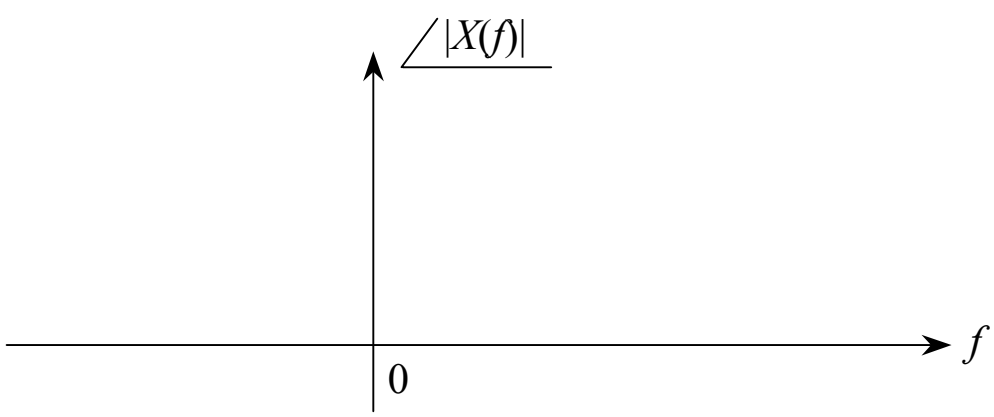
$$\mathrm{Re}\left[X(-f)\right] = \mathrm{Re}\left[X(f)\right]$$

$$\mathrm{Im}\left[X(-f)\right] = -\mathrm{Im}\left[X(f)\right]$$

$$\left|X(-f)\right| = \left|X(f)\right|$$

$$\angle X(-f) = -\angle X(f)$$





- Example

$$x(t) = \begin{cases} \cos 3t & -2 < t < 2 \\ 0 & ow \end{cases}$$

III. Discrete-Time Fourier Transform (DTFT)

Discrete-time signal: $x[n] = \mathrm{x}(nT_s)$

1) Sampling issues

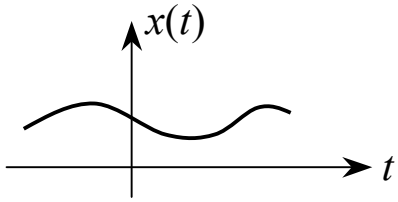
- <u>Sampling Theorem</u>:

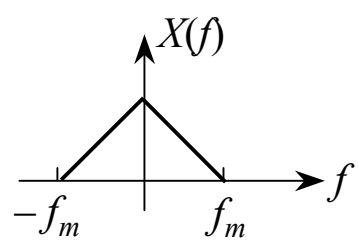
  When $x(t)$ is bandlimited with $X(f) = 0$ for $|f| > f_m$, then $x(t)$ is uniquely determined by its samples $x(nT_s)$ if

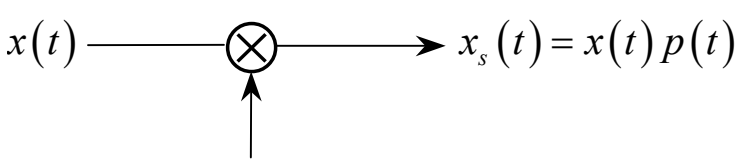
  $$f_s > 2 f_m$$

  $f_N = 2 f_m$ is called the Nyquist rate

- <u>Sampling theorem viewed from the frequency side</u>

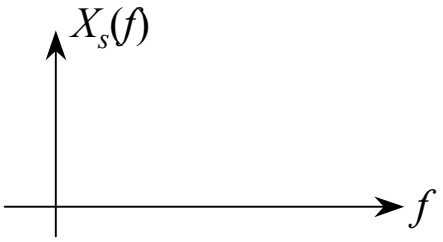






$$p(t) = \sum \delta(t - kT_s)$$

$$X_s(f) =$$



- <u>Aliasing</u>

- <u>Definition:</u> Normalized frequency

For $x(t) = \cos(2\pi f_0 t)$

$$\Longrightarrow x[n] = x(nT_s)$$
$$= \cos(2\pi f_0 n T_s)$$
$$=$$

- $X\left(e^{j2\pi f}\right) =$

2) Basic transform pair property

$$x[n] = e^{j2\pi f_0 n}$$

$$X\left(e^{j2\pi f}\right) =$$

3) Examples $x[n] = \cos(2\pi f_0 n T_s)$

$f_0 = 1/10Hz \quad T_s = 1s$

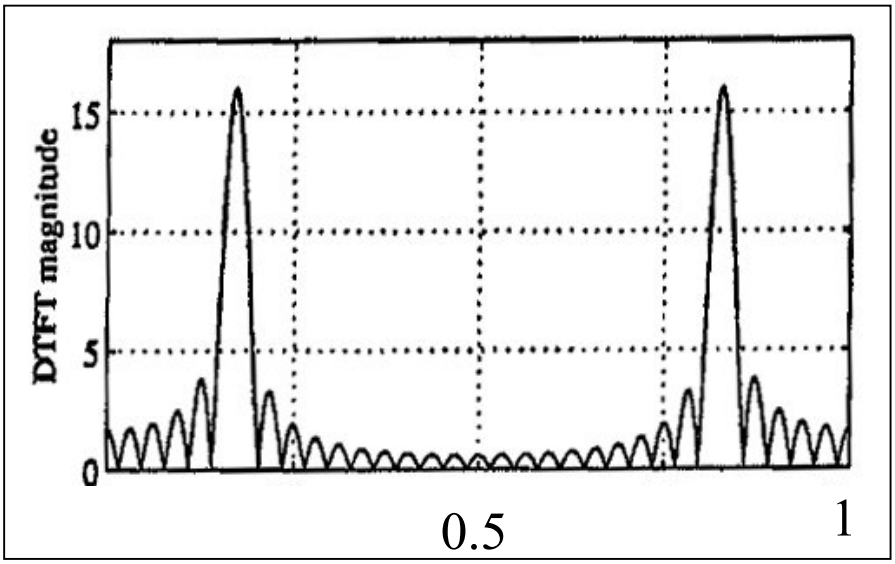
- Example:

$$x[n] = \cos\left(2\pi f_0 n T_s\right), \quad 0 \le n \le N-1$$

- <u>Example</u>: $x(t)=cos(2\pi f_0 n T_s)$

$N = 32$ $f_0 = 0.17$ Hz $T_s = 1$

($f_s = 1/T_s = 1$ Hz)



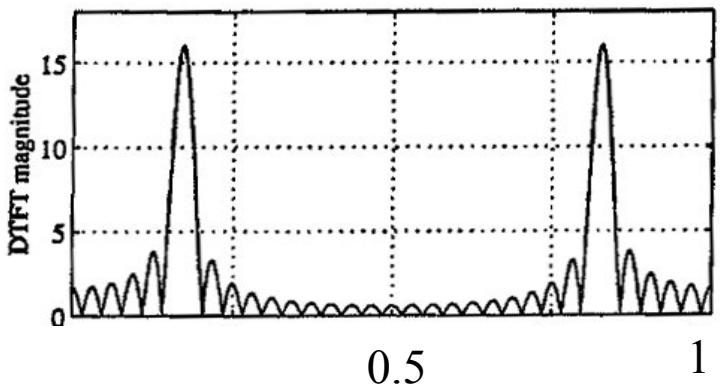
IV. Discrete Fourier Transform (DFT)

- Numerically we only have discrete values for f

- Definition: The R-point DFT $X(k)$ is defined as:

$$X(k) = X\left(e^{j2\pi f}\right)\Big|_{f=}$$

- **1) <u>DTFT</u>** $X(e^{j2\pi f})$ for $x[n] = \cos(2\pi f_0 n T_s) = \cos\left(\frac{2\pi f_0 n}{f_s}\right)$



$N = 32$
$f_0 = 11$ Hz; $f_s = 64$ Hz
$f_0 T_s =$

- **2) <u>R-point DFT</u>**
  - The DFT of x[n] is defined as:

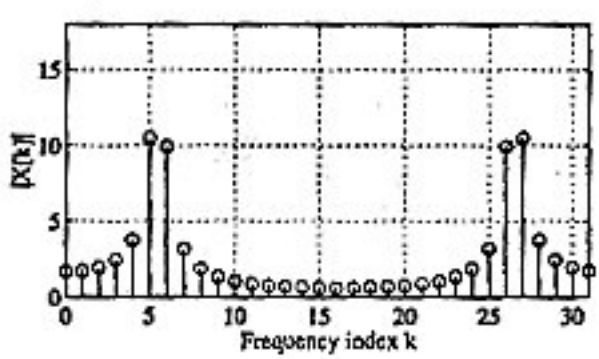
$$X[k] = X\left(e^{j2\pi f}\right)\Big|_{f=k/R,\ k=0,\cdots,R-1}$$

Frequency bin

  - X[k] is periodic with period =

  - DFT Example: assume $R = 32$ points

(a)



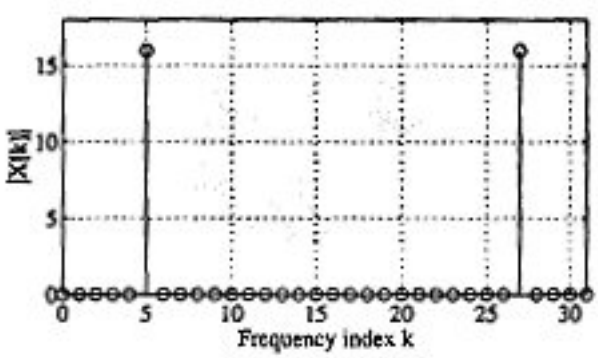
$N = 32$
$f_0 = 11$ Hz; $f_s = 64$ Hz

- DFT for $x[n] = \cos(2\pi f_0 n T_s) = \cos\left(\frac{2\pi f_0 n}{f_s}\right)$

$$X[k] = X\left(e^{j2\pi f}\right)\Big|_{f=k/R,\ k=0,\cdots,R-1}$$

Assume f_0 = 10 Hz; f_s = 64 Hz; $f_0 T_s$ = and R = 32 points

(b)



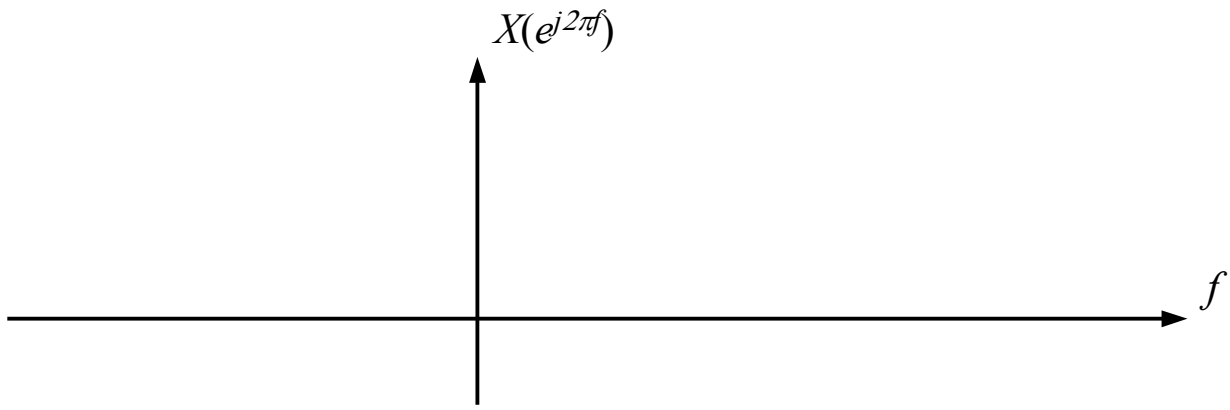
How do you explain the difference between plots (a) & (b) ?

- **3) <u>Varying the number of frequency bins *R*</u>**

Example: $x[n] = \frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right) + \cos\left(2\pi f_2 \frac{n}{f_s}\right);$

$$R = 16,\ f_s = 0.22\,Hz,\ f_2 = 0.34Hz,\ f_s = 1Hz$$

- Plot the DTFT $X(e^{j2\pi f})$

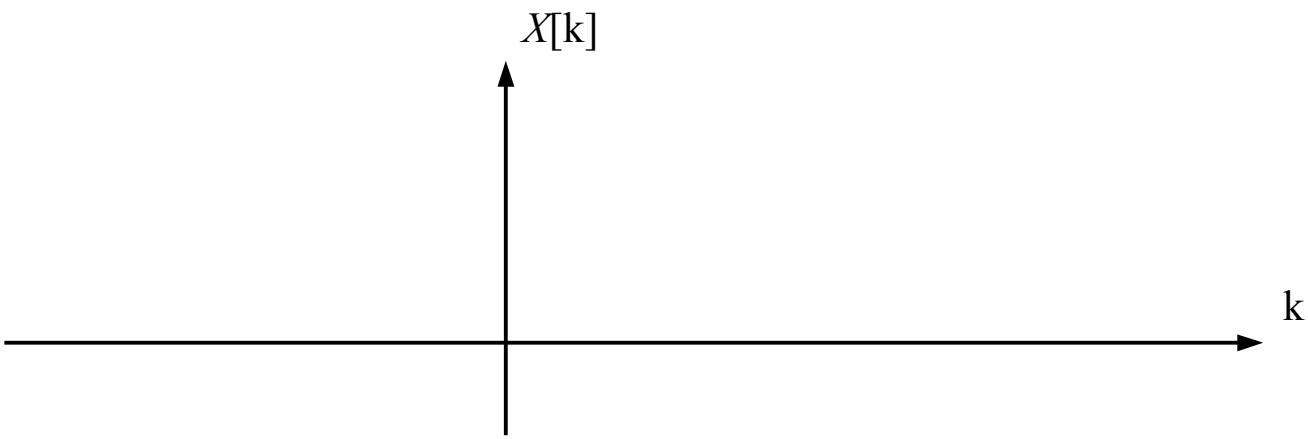


Example cont':

$$x[n] = \frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right) + \cos\left(2\pi f_2 \frac{n}{f_s}\right);$$

$$R = 16,\ f_s = 0.22\,Hz,\ f_2 = 0.34\,Hz,\ f_s = 1\,Hz$$

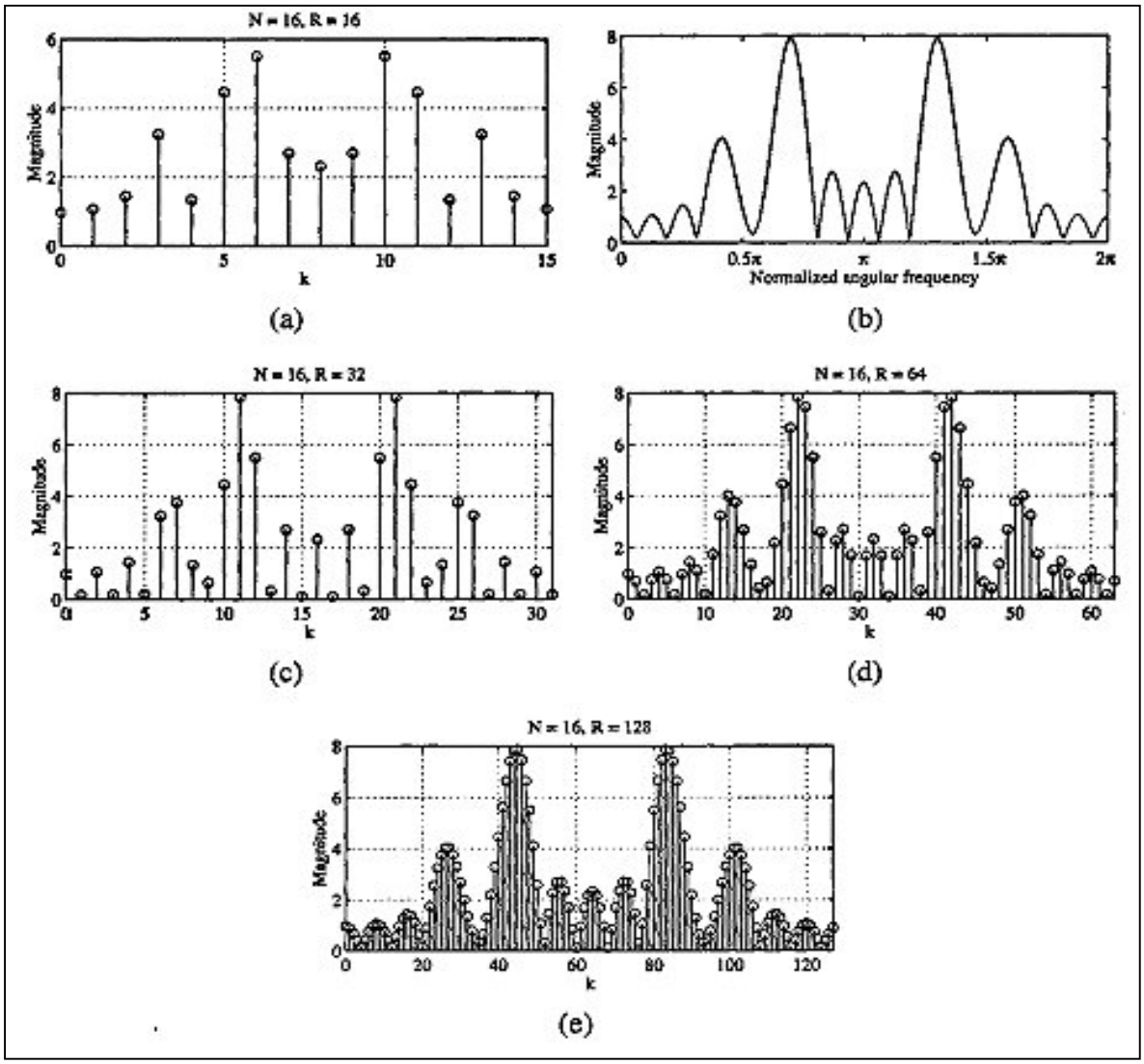
- Plot the DFT $X[\mathrm{k}]$



•**<u>Varying *R* Summary</u>;** R=[16,32,64,128]

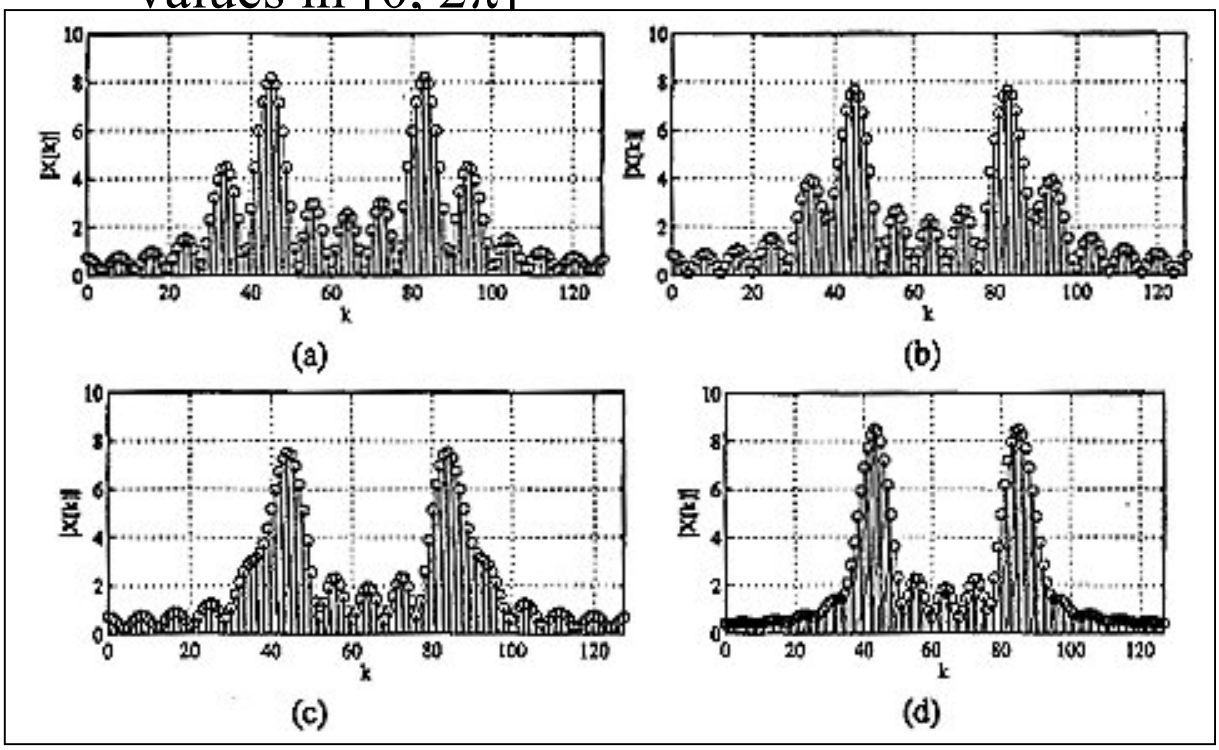
Example: $x[n] = \frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right) + \cos\left(2\pi f_2 \frac{n}{f_s}\right);$

$R = 16 \qquad f_s = 0.22 \qquad f_2 = 0.34$



- **4) <u>Resolution issues and DFT</u>**

$N = 16$; $R = 128$ (128 discrete frequencies values in $[0, 2\pi]$

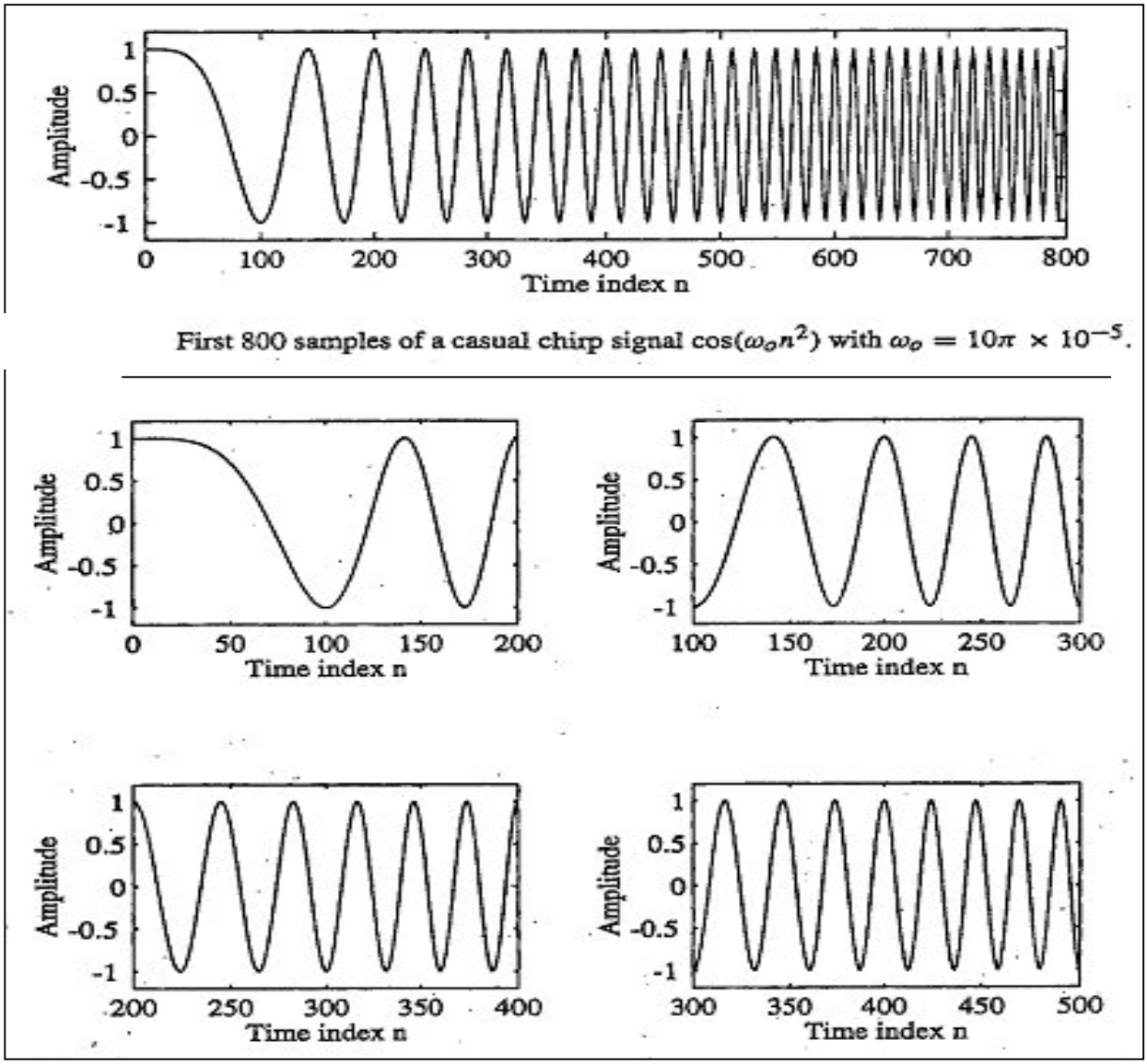


f_2 fixed, f_s=1, $f_2 = 0.34$ ==>k=

a) $f_1 = 0.28$ => k=
b) $f_1 = 0.29$ => k=
c) $f_1 = 0.30$ => k=
d) $f_1 = 0.31$ => k=

- **<u>Comments</u>:**

- **5) <u>Application of the DTFT to the Short-Time Fourier Transform</u>**

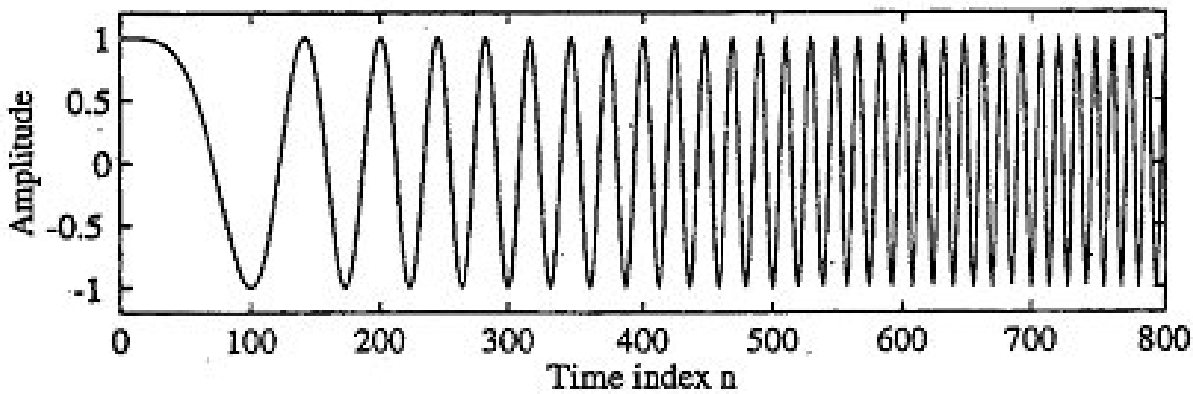


Various frames of length 200 samples extracted from the chirp signal

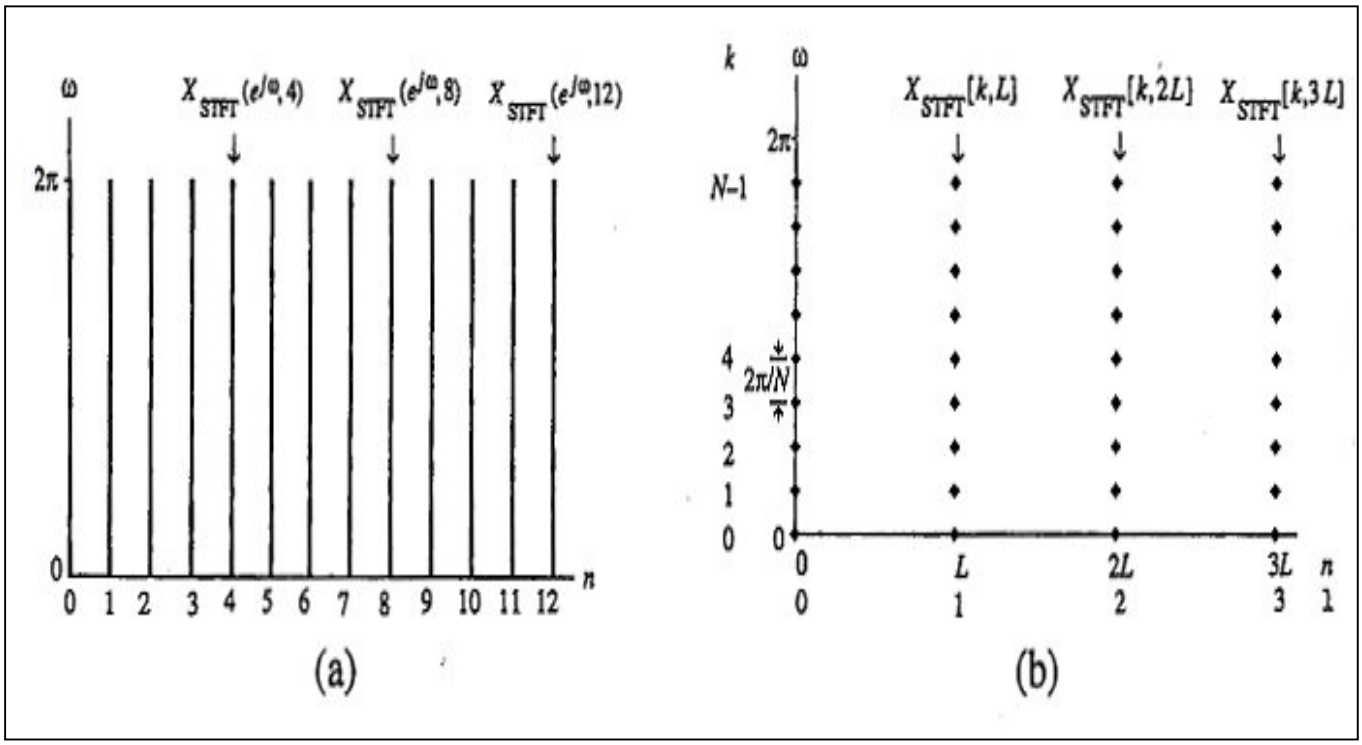
<u>Note</u>: Need to preserve the time-varying information of the frequency.

↓

How to do that?



First 800 samples of a casual chirp signal $\cos(\omega_o n^2)$ with $\omega_o = 10\pi \times 10^{-5}$.



Sampling grid in the (f,n)-plane for the sampled STFT XTFT[k,n], for N=9 & L=4