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SUPPLEMENTARY PROBLEMS FOR CHAPTER 6

1. A real signal s is observed in real additive Gaussian noise η. The observation x is given by

$$x = s + \eta$$

The noise has variance σ_η^2 and mean zero.

(a) First assume s is a deterministic but unknown signal. What is the maximum likelihood estimate for s?

(b) Now assume that s is a zero mean Gaussian random variable with variance σ_s^2. What is the MAP estimate for s?

2. The density function for a real Gaussian white noise process is given by

$$f_{x[n];\mathsf{P}}(\mathrm{x};\mathsf{P}) = \frac{1}{\sqrt{2\pi\mathsf{P}}} e^{-\frac{\mathrm{x}^2}{2\mathsf{P}}}$$

where P is the average power of the white noise process ($\mathsf{P} = \sigma_x^2$).

(a) Given N samples $x[0], x[1], \ldots, x[N-1]$ of the random process, find the maximum likelihood estimate for the power P.

(b) Show that this estimate is unbiased.

3. The density function for the *magnitude* $|x[n]|$ of a *complex* Gaussian white noise process is given by the Rayleigh density

$$f_{|x[n]|;\rho}(|\mathrm{x}|;\rho) = \frac{|\mathrm{x}|}{\rho} e^{-\frac{|\mathrm{x}|^2}{2\rho}}$$

where the parameter ρ is equal to one half the average power of the process ($\rho = \sigma_x^2/2$).

(a) Given N samples $x[0], x[1], \ldots, x[N-1]$ of the random process, find the maximum likelihood estimate for the power $\mathsf{P} = \sigma_x^2 = 2\rho$.

(b) Show that this estimate is unbiased.

4. Given observations $x[0], x[1], \ldots, x[N-1]$ of a Bernoulli process, find a maximum likelihood estimate for the parameter P and show that the estimate is unbiased. Is the estimate consistent?

   Hint: Let r be the number of $+1$'s in the observation sequence. Find the estimate first in terms of r and then express r in terms of the observations.

5. A Markov chain has the following transition matrix:

$$\mathbf{\Pi} = \begin{bmatrix} 0.6 & 0.6 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.4 \end{bmatrix}$$

   (a) Find the MAP estimate of the state at time $n+1$ given the state at time n. Provide your answer in a table such as the one below.

observation: state $s[n] =$	MAP estimate: $\hat{s}[n+1]$
1	
2	
3	
4	
5	

   (b) What would be the problem with using a mean-square estimate for the state?

6. A certain discrete random process consists of independent random variables described by the geometric distribution

$$\Pr[x[n] = r] = \begin{cases} \mathrm{P}(1-\mathrm{P})^{r-1} & r = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   for $0 < \mathrm{P} < 1$. Given observations $x[0], x[1], \ldots, x[N-1]$, find the maximum likelihood estimate of the parameter P.

7. A random process consists of *independent* identically distributed random variables with the Laplace density function

$$f_{x[n]}(x_n) = \frac{1}{\sqrt{2}\,\sigma} \exp\left[-\frac{\sqrt{2}\,|x_n - \mu|}{\sigma}\right]$$

Assume you have N samples $x[0], x[1], \ldots, x[N-1]$.

(a) Assume that the mean μ is known, i.e., $\mu = \mu_o$. Find the maximum likelihood estimate for the parameter σ (the standard deviation).

(b) Find the mean and variance of the estimate and show that the estimate is consistent.

(c) Now assume that the σ is known, i.e., $\sigma = \sigma_o$. What is the maximum likelihood estimate for μ? Is it unique? Show why or why not.

8. You are given K independent samples $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(K)}$ of a random vector $\boldsymbol{x}$.

(a) Assume first that the mean $\mathbf{m}_{\boldsymbol{x}}$ is known. Define an estimate of the covariance matrix as

$$\hat{\mathbf{C}}_{\boldsymbol{x}} = \frac{1}{K}\sum_{k=1}^{K}(\mathbf{x}^{(k)} - \mathbf{m}_{\boldsymbol{x}})(\mathbf{x}^{(k)} - \mathbf{m}_{\boldsymbol{x}})^{*T}$$

Show that this estimate is unbiased.

(b) Now define an estimate of the covariance matrix as

$$\hat{\mathbf{C}}'_{\boldsymbol{x}} = \frac{1}{K}\sum_{k=1}^{K}(\mathbf{x}^{(k)} - \hat{\mathbf{m}}_{\boldsymbol{x}})(\mathbf{x}^{(k)} - \hat{\mathbf{m}}_{\boldsymbol{x}})^{*T}$$

where $\hat{\mathbf{m}}_{\boldsymbol{x}}$ is the sample mean

$$\hat{\mathbf{m}}_{\boldsymbol{x}} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}^{(k)}$$

Show that this estimate is biased. Is it asymptotically unbiased?

(c) Finally modify the estimate in part (b) to

$$\hat{\mathbf{C}}''_{\boldsymbol{x}} = \frac{1}{K-1}\sum_{k=1}^{K}(\mathbf{x}^{(k)} - \hat{\mathbf{m}}_{\boldsymbol{x}})(\mathbf{x}^{(k)} - \hat{\mathbf{m}}_{\boldsymbol{x}})^{*T}$$

Show that this last estimate is unbiased.

9. In a certain image processing problem, the noise is multiplicative. Let x represent the measured intensity of a particular pixel in an image. This measured intensity is given by

$$x = \mathrm{s} \cdot w$$

where s is the true value of the image intensity ($\mathrm{s} > 0$ but unknown) and w is the noise, assumed to have an exponential probability density function

$$f_w(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & w < 0 \end{cases}$$

Assume N independent samples of x (i.e., N pixel values of the image) are observed all of which have the same true intensity s.

(a) Find the *maximum likelihood* estimate for s.

(b) Is the estimate consistent? State or show why or why not.

10. A certain random process has *independent samples.* Each sample $x[n]$ is described by the probability density function

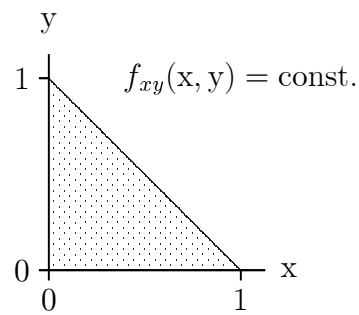
$$f_x(\mathrm{x}) = \beta \mathrm{x}^{\beta-1} \qquad 0 \leq \mathrm{x} \leq 1$$

The parameter β is unknown. You are given N samples of the process $x_n = x[n], \quad n = 1, 2, \ldots, N$ and want to estimate the parameter β.

(a) Write an expression for the likelihood function involving the parameter β and the samples $x_n, \quad n = 1, 2, \ldots, N$.

(b) Find the maximum likelihood estimate for β.
<u>Hint:</u> Consider using the *log* likelihood function in this part.

11. Two random variables x and y are uniformly distributed over the region shown below:



(a) Find an sketch the conditional density function $f_{y|x}(\mathrm{y}|\mathrm{x})$.

(b) Find an expression for the *mean-square* estimate of y in terms of x.