SUPPLEMENTARY PROBLEMS FOR CHAPTER 3

1. A certain binary-valued random process can be described as a Markov chain with states 1 and 2 and transition matrix

$$\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

   (a) Assume the process starts at $n = 0$ and that the initial state probabilities are equally likely, i.e., $\mathrm{p}_1[0] = \mathrm{p}_2[0] = \frac{1}{2}$ What is the probability of starting in state 1 and remaining in that state for the next five transitions?

   (b) Now assume that the process has been going on for a long time. If you observe the process at some randomly selected time, what is the probability that the process will be in state 1 and remain in that state for the next five transitions?

   (c) What are the third order state transition probabilities, i.e., the probability that the process will be in state j at point n given that it was in state i at point $n - 3$?

2. Tell if the following statements are correct or incorrect. If incorrect, tell why.

   (a) A random walk is always a stationary random process.

   (b) A Bernoulli process normally has independent increments.

   (c) Processes with independent increments are always stationary.

   (d) Strictly ergodic processes must have zero mean.

   (e) If a process is described by only two random parameters then it must be periodic.

3. A random process is defined as follows:

$$x[n] = \begin{cases} 1 & \text{with probability } 0.6 \\ -1 & \text{with probability } 0.4 \end{cases}$$

   if $x[n-1] = 1$,

$$x[n] = \begin{cases} 1 & \text{with probability } 0.4 \\ -1 & \text{with probability } 0.6 \end{cases}$$

   if $x[n-1] = -1$.

(a) Given that $x[n-2] = 1$, what are the probabilities that $x[n] = 1$ and $x[n] = -1$?

(b) Given that $x[n-2] = -1$, what are the probabilities that $x[n] = 1$ and $x[n] = -1$?

4. A *counting process* is defined as

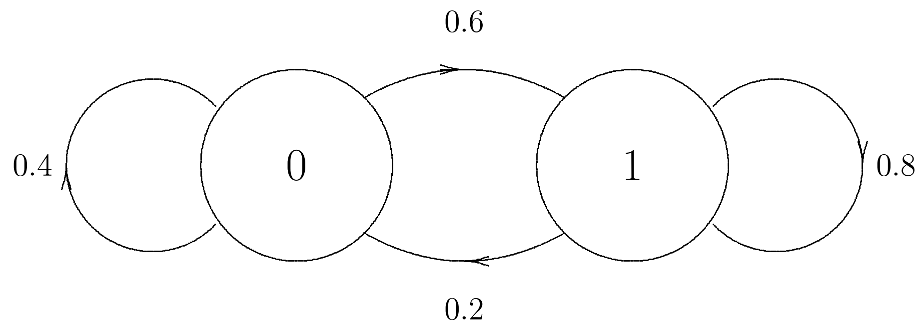
$$y[n] = \sum_{k=1}^{n} \tfrac{1}{2}\left(x[k] + 1\right)$$

where $x[n]$ is a Bernoulli process with parameter P. (The Bernoulli process is assumed to take on values of $+1$ and -1.)

(a) Find the mean (expected value) of the counting process.

(b) Is the process $y[n]$ stationary? State why or why not.

5. A certain binary sequence is described by a Markov chain with state diagram shown below:



(a) Given that the process starts in state "1", what is the probability that the process will be in state "1" after *two* transitions? Again, given that it starts in state "1", what is the probability that it will be in state "0" after two transitions?

(b) Now assume the process has been going on for a long time. Find the probability of observing N consecutive symbols all of the same type. Express your answer as a function of N.

6. A binary data sequence $x[n]$ consisting of 0's and 1's is modeled by a two-state Markov chain with transition matrix

$$\mathbf{\Pi} = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$$

(The first row or column of the matrix represents state 0; the second row or column represents state 1.)

(a) Given that you have just observed a 1 at time n_o, what is the probability that the *next* 1 occurs at time $n_o + 4$?

(b) Given that you have just observed a 1 at time n_o, what is the probability that $x[n_o + 4] = 1$?

7. A Markov chain has the following transition matrix:

$$\mathbf{\Pi} = \begin{bmatrix} 0.6 & 0.6 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.4 \end{bmatrix}$$

(a) Draw the transition diagram for the process.

(b) Assume the process has been going on for a long time. Find the limiting-state probabilities of each state.

8. A certain random process taking on vales of $+1$ and -1 can be modeled as a Markov chain. The transition matrix for the process is

$$\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

(a) Show that after observing the process for a sufficiently long time, the probability of a $+1$ and a -1 are equal.

(b) What is the probability of a sequence of 10 consecutive $+1$'s? Is this more or less likely than the occurance of a sequence of 10 consecutive $+1$'s in a binary white noise process?

(c) What is the probability of a sequence of 10 consecutive values alternating in sign? Is this more or less likely than the probability of such a sequence of values in a binary white noise process?

9. Consider a random process consisting of *independent* random variables $\xi[n]$ such that

$$\xi[n] = \begin{cases} +1 & \text{with probability P} \\ -1 & \text{with probability Q} \\ 0 & \text{with probability } 1 - \text{P} - \text{Q} \end{cases}$$

(a) Calculate the mean and variance of the process. Is $\xi[n]$ stationary?

(b) Now consider the process

$$x[n] = \sum_{k=1}^{n} \xi[k]\,; \qquad n > 0$$

What is the mean of the process $\mathcal{E}\{x[n]\}$? What is the condition on P and Q that you can impose so that the process $x[n]$ would have zero mean?

(c) Is there any condition on P and Q that you can impose so that the process $x[n]$ would be stationary?

(d) Can $x[n]$ be represented by a Markov chain? If so, draw the state transition diagram.

10. A random process $y[n]$ is defined by

$$y[n] = \tfrac{1}{2}\left(x[n] + x[n-1]\right)$$

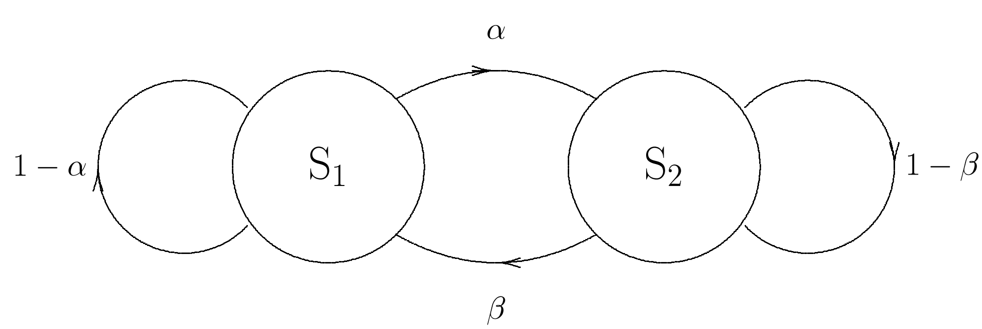
where $x[n]$ is a Bernoulli process with parameter P.

(a) What is the mean and the variance of the process?

(b) What is the distribution of the process (i.e., the probability that $y[n]$ equals various numbers).

(c) Does the process have independent increments? Show why or why not.

11. A Markov chain has the state diagram illustrated below:



(a) Suppose that $\beta = 1/3$. What value of α would be necessary so that the limiting state probabilities are given by $\bar{p}_1 = 2/3$ and $\bar{p}_2 = 1/3$?

(b) Again suppose that α is unknown and β is equal to 1/3. What values of the limiting state probability $\bar{\mathrm{p}}_1$ are *possible*?

12. Assume that $x[n]$ is a binary white noise process and $y[n]$ is defined as

$$y[n] = x[n] - x[n-1]$$

Answer the following questions about $y[n]$:

(a) Is $y[n]$ stationary?

(b) Is $y[n]$ ergodic?

(c) Is the mean of $y[n]$ equal to zero?

(d) What is the variance of $y[n]$?