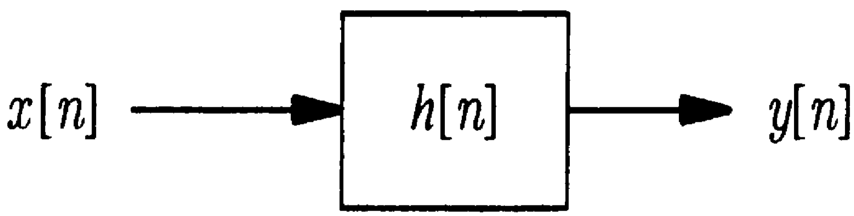
TRANSFORMATION BY LINEAR SYSTEMS



$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k]
\end{aligned}
$$

MEAN

$$\mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\} \quad \text{or...} \quad \boxed{m_y = m_x \cdot \sum_{k=-\infty}^{\infty} h[k]}$$

TRANSFORMATION BY LINEAR SYSTEMS (cont'd.)

CORRELATION AND CROSS-CORRELATION FUNCTIONS

$$\mathcal{E}\{y[n]y^*[n-l]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]y^*[n-l]\}$$

$$R_y[l] = \sum_{k=-\infty}^{\infty} h[k]R_{xy}[l-k] \quad \text{or}\ldots \quad \boxed{R_y[l] = h[l] * R_{xy}[l]}$$

Similarly, by multiplying by $x^*[n-l]$ and taking the expectation ...

$$\boxed{R_{yx}[l] = h[l] * R_x[l]}$$

TRANSFORMATION BY LINEAR SYSTEMS (cont'd.)

COVARIANCE FUNCTIONS

$$R_y[l] = \sum_{k=-\infty}^{\infty} h[k] R_{xy}[l-k] \qquad m_y = m_x \sum_{k=-\infty}^{\infty} h[k]$$

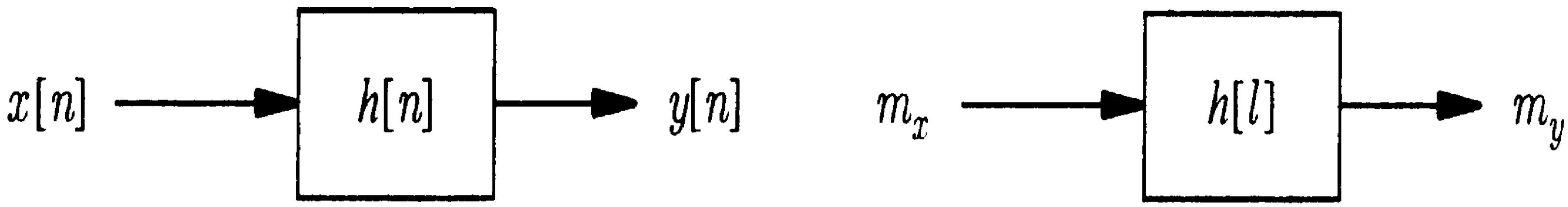
$$C_y[l] \;=\; R_y[l] \;-\; m_y m_y^*$$

$$C_y[l] = \sum_{k=-\infty}^{\infty} h[k] \cdot \left(R_{xy}[l-k] - m_x m_y^* \right) = \sum_{k=-\infty}^{\infty} h[k] C_{xy}[l-k]$$

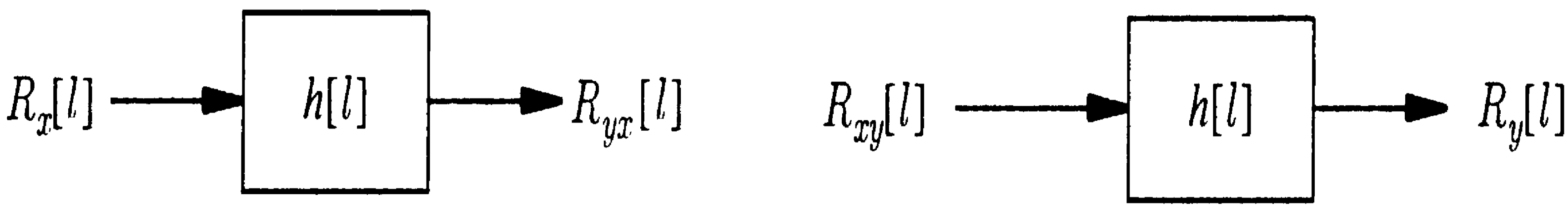
or . . . $\boxed{C_y[l] = h[l] * C_{xy}[l]}$

and similarly . . . $\boxed{C_{yx}[l] = h[l] * C_x[l]}$

TRANSFORMATIONS: MNEMONIC DEVICE



$$R_{xy}[l] = R^*_{yx}[-l]$$



Insert Example 5.1 here.

COMBINED RELATIONS

Start with

$$R_{yx}[l] = h[l] * R_x[l] \quad \Rightarrow \quad R^*_{yx}[-l] = h^*[-l] * R^*_x[-l]$$

or $$R_{xy}[l] = h^*[-l] * R_x[l]$$

Then

$$R_y[l] = h[l] * R_{xy}[l] = h[l] * h^*[-l] * R_x[l]$$

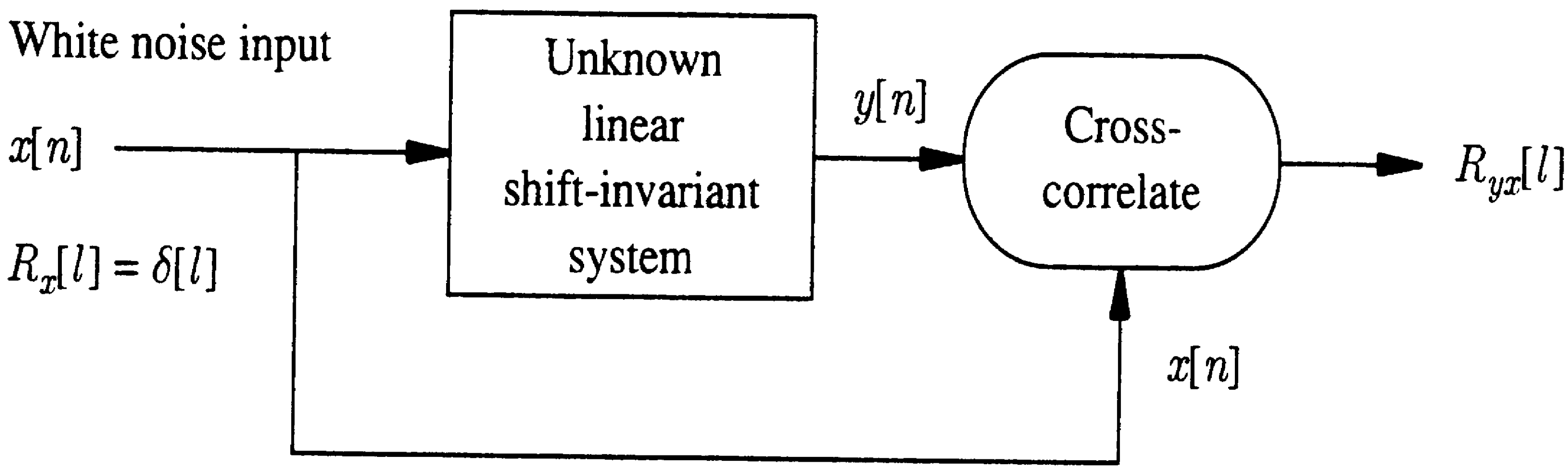
LINEAR TRANSFORMATION SUMMARY

System defined by: $y[n] = h[n] * x[n]$

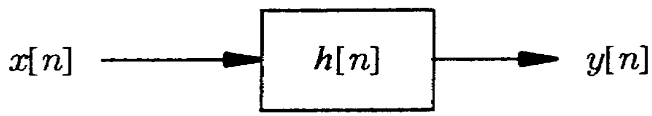
$R_{yx}[l]{=}h[l]{*}R_x[l]$	$S_{yx}(e^{\jmath\omega}){=}H(e^{\jmath\omega})S_x(e^{\jmath\omega})$	$S_{yx}(z){=}H(z)S_x(z)$
$R_{xy}[l]{=}h^*[-l]{*}R_x[l]$	$S_{xy}(e^{\jmath\omega}){=}H^*(e^{\jmath\omega})S_x(e^{\jmath\omega})$	$S_{xy}(z){=}H^*(1/z^*)S_x(z)$
$R_y[l]{=}h[l]{*}R_{xy}[l]$	$S_y(e^{\jmath\omega}){=}H(e^{\jmath\omega})S_{xy}(e^{\jmath\omega})$	$S_y(z){=}H(z)S_{xy}(z)$
$R_y[l]{=}h[l]{*}h^*[-l]{*}R_x[l]$	$S_y(e^{\jmath\omega}){=}\|H(e^{\jmath\omega})\|^2S_x(e^{\jmath\omega})$	$S_y(z){=}H(z)H^*(1/z^*)S_x(z)$

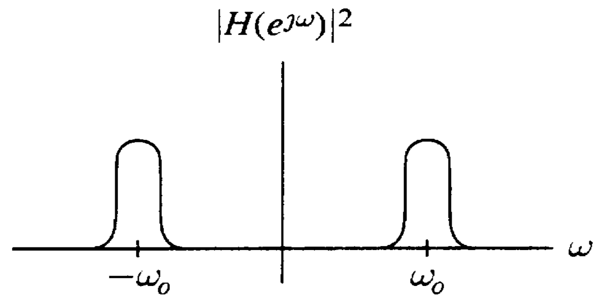
APPLICATION: SYSTEM IDENTIFICATION

$$R_{yx}[l] = h[l] * \delta[l] = h[l]$$



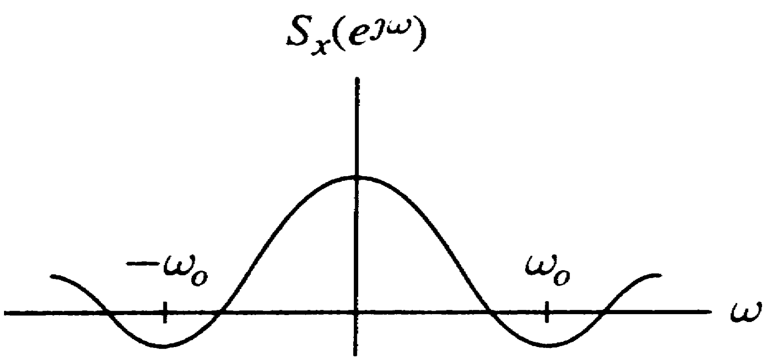
APPLICATION: PROOF THAT $S_x(e^{j\omega}) \geq 0$

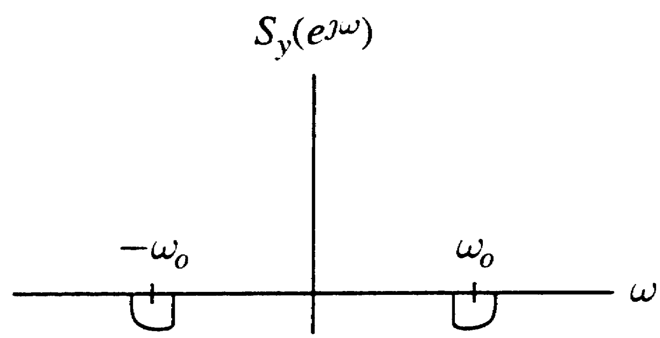




$$S_y(e^{j\omega}) = S_x(e^{j\omega}) \left| H(e^{j\omega}) \right|^2$$

$$\text{Avg. power} = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_y(e^{j\omega}) d\omega$$





SYSTEM IN DIFFERENCE EQUATION FORM

LINEAR SYSTEM DESCRIPTION

$$y[n] + a_1 y[n-1] + \cdots + a_P y[n-P] = b_0 x[n] + \cdots + b_Q x[n-Q]$$

Take expectation to obtain...

EQUATION FOR THE MEAN

$$m_y + a_1 m_y + \cdots + a_P m_y = b_0 m_x + \cdots + b_Q m_x$$

or... $$\boxed{m_y = \frac{\sum_{j=0}^{Q} b_j}{1 + \sum_{i=1}^{P} a_i}\, m_x}$$

DIFFERENCE EQUATION FORM (cont'd.)

Starting with . . .

$$y[n] + a_1 y[n-1] + \cdots + a_P y[n-P] = b_0 x[n] + \cdots + b_Q x[n-Q]$$

Multiply by $y^*[n-l]$ or $x^*[n-l]$ and take expectation to obtain . . .

CORRELATION DIFFERENCE EQUATIONS

$$R_y[l] + a_1 R_y[l-1] + \cdots + a_P R_y[l-P] = b_0 R_{xy}[l] + \cdots + b_Q R_{xy}[l-Q]$$

$$R_{yx}[l] + a_1 R_{yx}[l-1] + \cdots + a_P R_{yx}[l-P] = b_0 R_x[l] + \cdots + b_Q R_x[l-Q]$$

DIFFERENCE EQUATION FORM (cont'd.)

To find equation for the covariance, begin with

$$y[n] + a_1 y[n-1] + \cdots + a_P y[n-P] = b_0 x[n] + \cdots + b_Q x[n-Q]$$

and subtract

$$m_y + a_1 m_y + \cdots + a_P m_y = b_0 m_x + \ \cdots \ + b_Q m_x$$

to obtain

$$(y[n]-m_y)+a_1(y[n-1]-m_y)+\cdots+a_P(y[n-P]-m_y)$$

$$= b_0(x[n]-m_x) + \ \cdots \ + b_Q\,(x[n-Q]-m_x)$$

DIFFERENCE EQUATION FORM (cont'd.)

Now starting with

$$(y[n]-m_y)+a_1(y[n-1]-m_y)+\cdots+a_P(y[n-P]-m_y)$$

$$= b_0(x[n]-m_x)+ \cdots + b_Q(x[n-Q]-m_x)$$

Multiply by $(y[n-l]-m_y)^*$ or $(x[n-l]-m_x)^*$ and take expectation to obtain . . .

COVARIANCE DIFFERENCE EQUATIONS

$$C_y[l]+a_1C_y[l-1]+\cdots+a_PC_y[l-P] = b_0C_{xy}[l]+\cdots+b_QC_{xy}[l-Q]$$

$$C_{yx}[l]+a_1C_{yx}[l-1]+\cdots+a_PC_{yx}[l-P] = b_0C_x[l]+\cdots+b_QC_x[l-Q]$$

MINIMUM-PHASE DEFINITIONS

MINIMUM-PHASE POLYNOMIAL

Has all roots *inside* the unit circle.

MINIMUM-PHASE TRANSFER FUNCTION

Ratio of two minimum-phase polynomials:

$$H_{min}(z) = \frac{B(z)}{A(z)} \quad \text{with } A(z) \text{ and } B(z) \text{ minimum-phase}$$

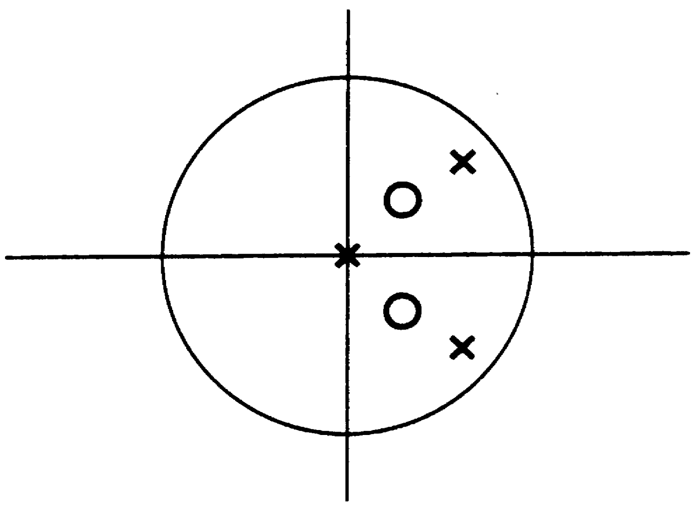
MINIMUM-PHASE SYSTEM

Causal system with minimum-phase transfer function.

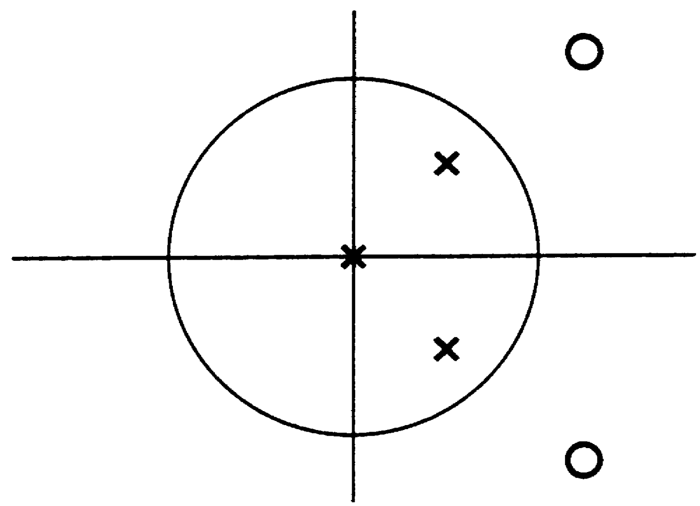
- It is a *causal stable system with a causal stable inverse.*

MINIMUM- AND NONMINIMUM-PHASE SYSTEMS

MINIMUM-PHASE



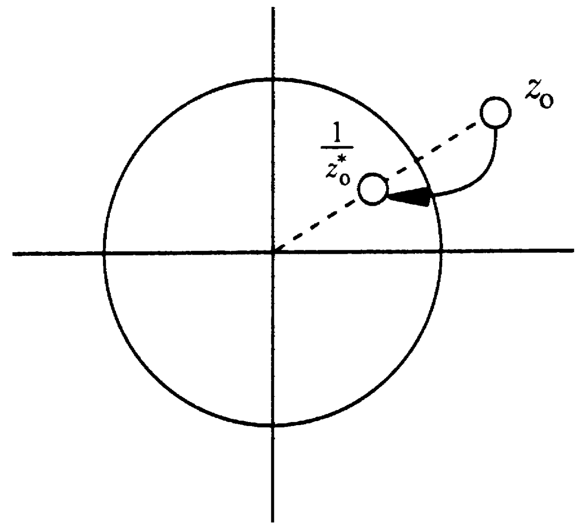
NONMINIMUM-PHASE



MOVING A ZERO TO INSIDE THE UNIT CIRCLE

Multiply $H(z)$ by

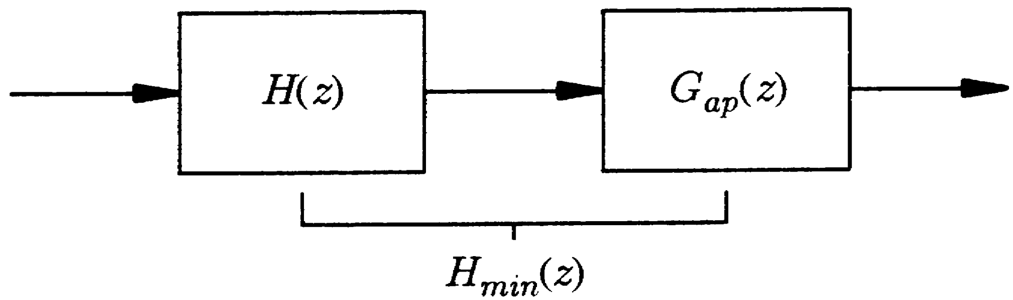
$$\frac{1 - z_o^* z}{z - z_o} = \frac{z^{-1} - z_o^*}{1 - z_o z^{-1}}$$



This does not change the magnitude of $H(z)$.

$$\left| \frac{z^{-1} - z_o^*}{1 - z_o z^{-1}} \right|_{z=e^{j\omega}} = \frac{e^{-j\omega} - z_o^*}{1 - z_o e^{-j\omega}} = e^{-j\omega} \cdot \underbrace{\frac{1 - z_o^* e^{+j\omega}}{1 - z_o e^{-j\omega}}}_{\text{magnitude=1}}$$

CONVERSION OF A SYSTEM TO MINIMUM-PHASE



$G_{ap}(z)$ consists of a product of terms of the form

$$\frac{z^{-1} - z_i^*}{1 - z_i z^{-1}} \qquad \text{or} \qquad \frac{1 - z_i z^{-1}}{z^{-1} - z_i^*}$$

Since these terms have magnitude 1 for all frequencies,

$$|H_{min}(e^{\jmath\omega})| = |H(e^{\jmath\omega})|$$

PALEY-WIENER CONDITION

A complex spectral density function satisfying the Payley-Wiener condition

$$\int_{-\pi}^{\pi} |\ln S_x(e^{\jmath\omega})| d\omega < \infty$$

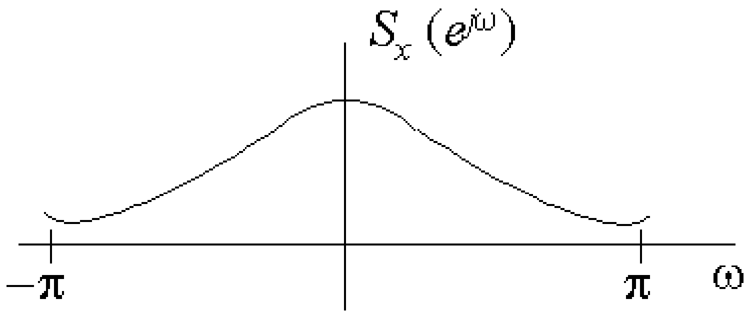
can be factored as

$$S_x(z) = \mathcal{K}_{\mathrm{o}} \cdot H_{ca}(z) H^*_{ca}(1/z^*)$$

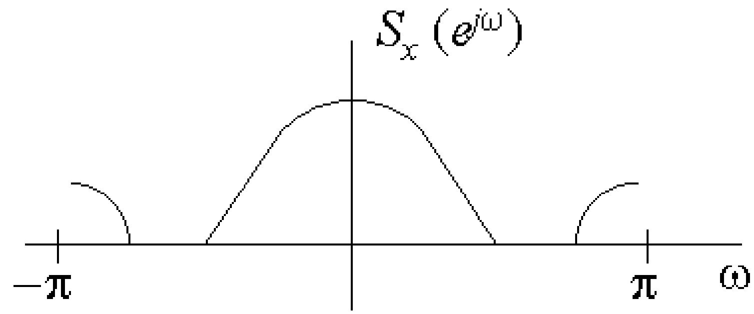
where $H_{ca}(z)$ is a causal stable system with a causal stable inverse.

- The Payley-Wiener condition is both *necessary and sufficient.*
- A process satisfying the Payley-Wiener condition is said to be a *regular* process.

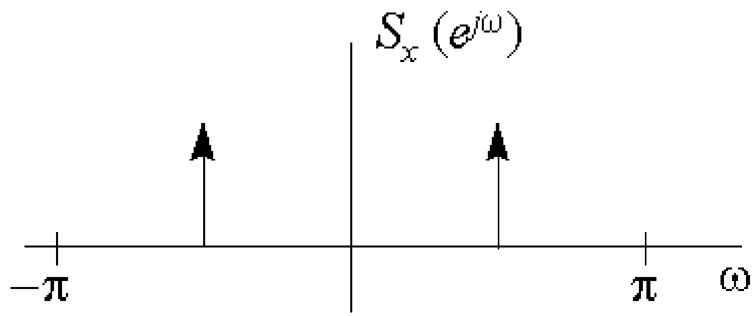
PALEY-WIENER EXAMPLES



satisfies Paley-Wiener



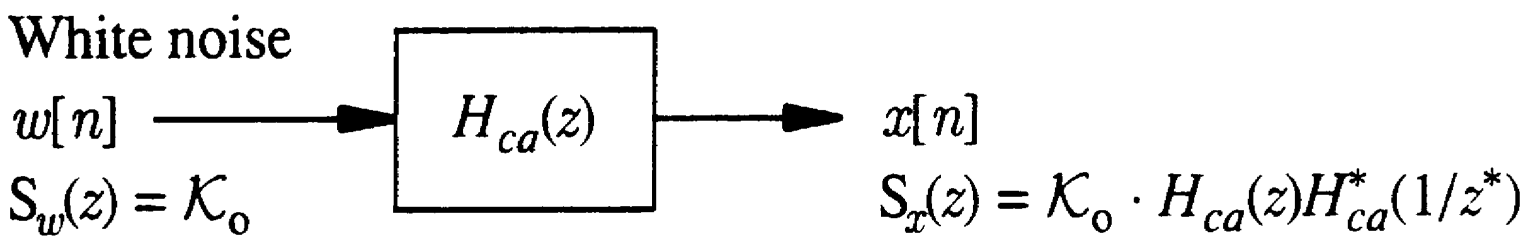
does *not* satisfy Paley-Wiener



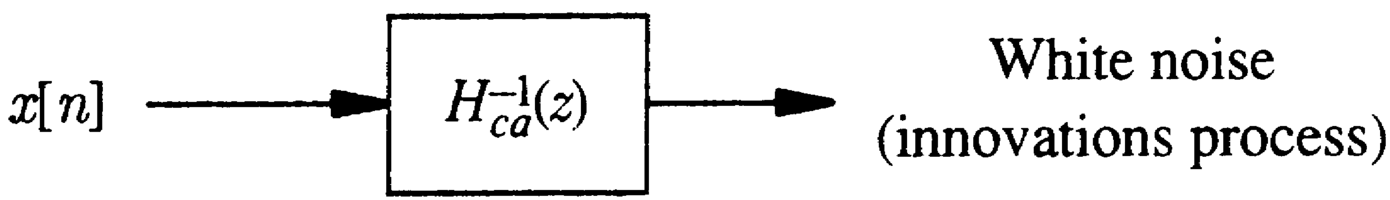
does *not* satisfy Paley-Wiener

INNOVATIONS REPRESENTATION FOR A REGULAR PROCESS

MODEL FOR THE PROCESS



INVERSE FILTER



Insert Example 5.5 here.

SPECTRAL FACTORIZATION FOR RATIONAL POLYNOMIALS

- All finite rational $S_x(z)$ satisfy Paley-Wiener and are thus factorable:

$$S_x(z) = \mathcal{K}_o \cdot H_{ca}(z) H_{ca}^*(1/z^*)$$

- The term $H_{ca}(z)$ represents a *minimum-phase* system.

- By convention, the numerator and denominator of $H_{ca}(z)$ are *comonic* polynomials in z^{-1}.

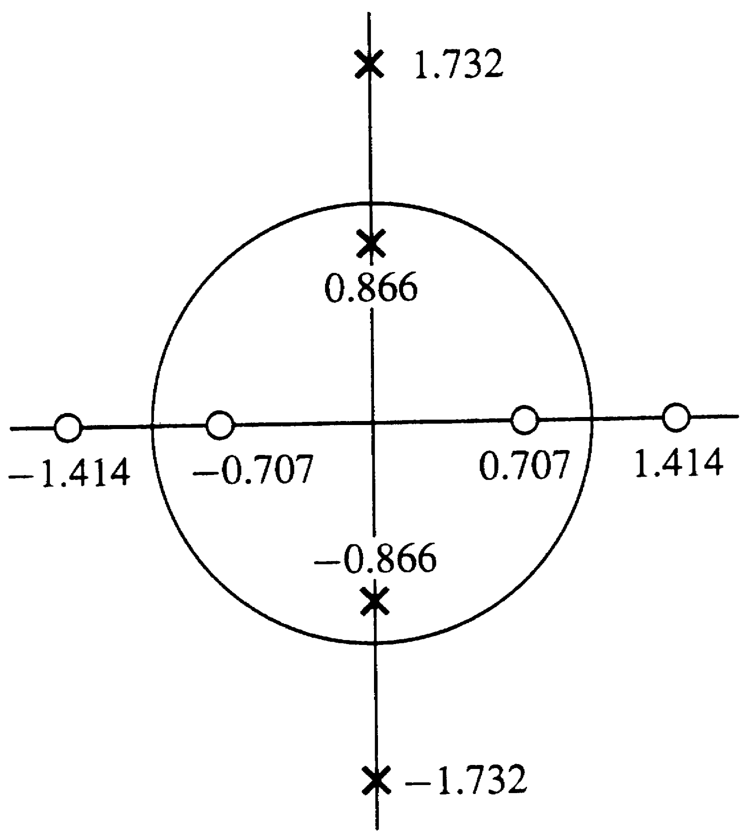
SPECTRAL FACTORIZATION FOR RATIONAL $S_x(z)$

Since $S_x(z) = S_x^*(1/z^*)$ poles and zeros occur in <u>*conjugate reciprocal*</u> locations and $S_x(z)$ can be factored as:

$$S_x(z) = \mathcal{K}_\mathsf{o} \cdot \underbrace{\frac{(1 - z_o z^{-1})(\quad)\cdots(\quad)}{(\quad)(\quad)\cdots(\quad)}}_{H_{ca}(z)} \cdot \underbrace{\frac{(1 - z_o^* z)(\quad)\cdots(\quad)}{(\quad)(\quad)\cdots(\quad)}}_{H_{ca}^*(1/z^*)}$$

- $H_{ca}(z)$ is the ratio of two <u>*comonic*</u> polynomials.

SPECTRAL FACTORIZATION ILLUSTRATION FOR RATIONAL POLYNOMIALS



$$S_x(z) = \frac{-12z^2+30-12z^{-2}}{6z^2+20+6z^{-2}}$$

$$\begin{aligned} H_{ca}(z) &= \frac{(1+0.707z^{-1})(1-0.707z^{-1})}{(1+\jmath 0.866z^{-1})(1-\jmath 0.866z^{-1})} \\ &= \frac{1-0.5z^{-2}}{1+0.333z^{-2}} \end{aligned}$$

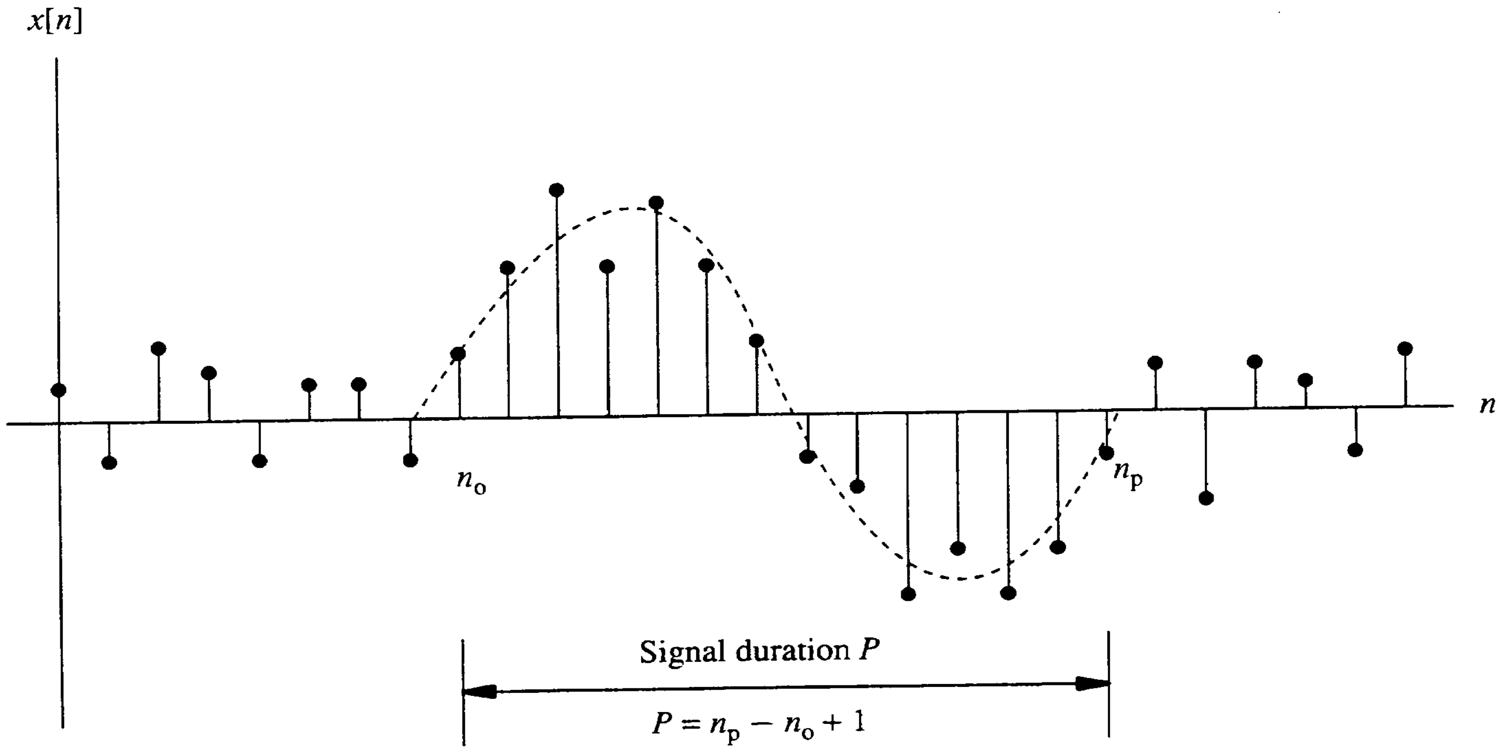
Insert Example 5.101 here.

Insert Example 5.6 here.

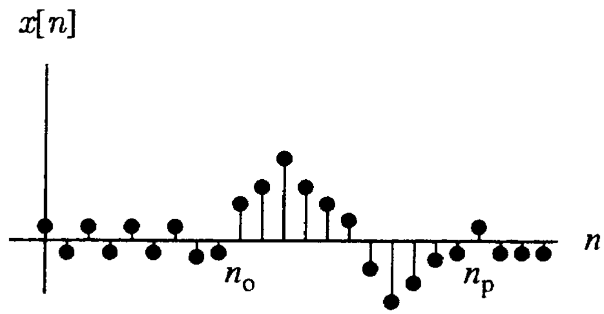
MATCHED FILTER

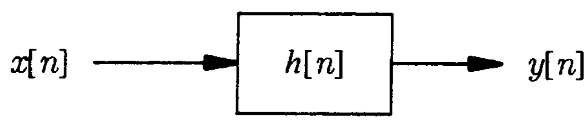
- Used for detection of signal in additive noise.
  - decide: signal present or not present
  - estimate the arrival time
- Applications are in radar, sonar, communications.
- Developed as a type of “optimal filtering” problem.

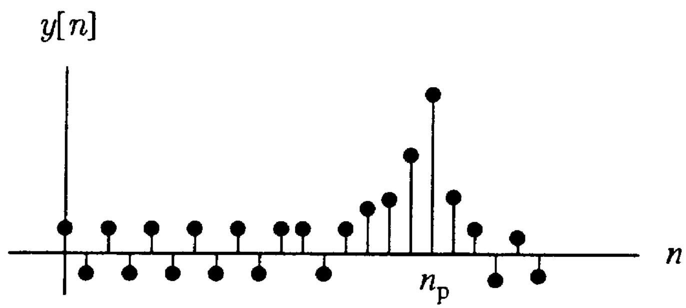
SIGNAL IN ADDITIVE NOISE



MATCHED FILTER RESPONSE







PROBLEM STATEMENT

RECEIVED SIGNAL

$$x[n] = \mathsf{s}[n] + \eta[n]$$

PROCESSED SIGNAL

$$y[n] = y_{\mathsf{s}}[n] + y_{\eta}[n]$$

OUTPUT SIGNAL-TO-NOISE RATIO

$$\mathsf{SNR} \stackrel{\mathsf{def}}{=} \frac{|y_{\mathsf{s}}[n_{\mathsf{p}}]|^2}{\mathcal{E}\left\{|y_{\eta}[n_{\mathsf{p}}]|^2\right\}}$$

FILTER EQUATIONS

$$
\begin{aligned}
y[n_{\mathsf{p}}] &= \sum_{k=0}^{P} h[k]x[n_{\mathsf{p}} - k] \\
&= \mathbf{h}^T \tilde{\boldsymbol{x}}
\end{aligned}
$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

where

$$
\mathbf{h} = \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[P-1] \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x[n_{\mathsf{o}}] \\ x[n_{\mathsf{o}} + 1] \\ \vdots \\ x[n_{\mathsf{p}}] \end{bmatrix}
$$

FORMULATION OF SNR

Filter output is $\qquad y[n_\mathsf{p}] = y_\mathsf{s}[n_\mathsf{p}] + y_\eta[n_\mathsf{p}]$

where $\qquad y_\mathsf{s}[n_\mathsf{p}] = \mathbf{h}^T\tilde{\mathbf{s}} = \tilde{\mathbf{s}}^T\mathbf{h} \qquad y_\eta[n_\mathsf{p}] = \mathbf{h}^T\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}^T\mathbf{h}$

Then

$$|y_\mathsf{s}[n_\mathsf{p}]|^2 = (\mathbf{h}^T\tilde{\mathbf{s}})^*(\tilde{\mathbf{s}}^T\mathbf{h}) = \mathbf{h}^{*T}\tilde{\mathbf{s}}^*\tilde{\mathbf{s}}^T\mathbf{h}$$

$$\mathcal{E}\left\{|y_\eta[n_\mathsf{p}]|^2\right\} = \mathcal{E}\left\{(\mathbf{h}^T\tilde{\boldsymbol{\eta}})^*(\tilde{\boldsymbol{\eta}}^T\mathbf{h})\right\} = \mathbf{h}^{*T}\tilde{\mathbf{R}}^*_{\boldsymbol{\eta}}\mathbf{h} = \mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}$$

therefore . . .

$$\mathsf{SNR} = \frac{|y_\mathsf{s}[n_\mathsf{p}]|^2}{\mathcal{E}\left\{|y_\eta[n_\mathsf{p}]|^2\right\}} = \frac{\mathbf{h}^{*T}\tilde{\mathbf{s}}^*\tilde{\mathbf{s}}^T\mathbf{h}}{\mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}}$$

OPTIMIZATION PROBLEM

Maximize

$$\mathsf{SNR} = \frac{\mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h}}{\mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}}$$

subject to

$$\mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h} = 1$$

Form the LaGrangian

$$\mathcal{L} = \mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h} + \lambda(1 - \mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h})$$

OPTIMIZATION SOLUTION

To find a stationary point of

$$\mathcal{L} = \mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h} + \lambda(1 - \mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h})$$

require

$$\nabla_{\mathbf{h}^*}\mathcal{L} = \tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h} - \lambda\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h} = \mathbf{0} \quad \Rightarrow \quad (\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T})\mathbf{h} = \lambda\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}$$

The solution implies that

$$\mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h} = \lambda\mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}$$

or . . .

$$\lambda = \frac{\mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h}}{\mathbf{h}^{*T}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}} = \mathsf{SNR}$$

OPTIMIZATION SOLUTION (cont'd.)

To solve

$$(\tilde{\mathbf{s}}^*\tilde{\mathbf{s}}^T)\mathbf{h} = \lambda \mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}$$

note that:

- There are $P-1$ solutions $\tilde{\mathbf{s}}^T\mathbf{h} = 0$ yielding $\lambda = 0$.

- The desired solution has $\mathbf{h} \propto \mathbf{R}_{\eta}^{-1}\tilde{\mathbf{s}}^*$:

$$\tilde{\mathbf{s}}^* \underbrace{\tilde{\mathbf{s}}^T\mathbf{R}_{\boldsymbol{\eta}}^{-1}\tilde{\mathbf{s}}^*}_{\text{scalar} = \lambda} = \lambda \mathbf{R}_{\boldsymbol{\eta}}\mathbf{R}_{\boldsymbol{\eta}}^{-1}\tilde{\mathbf{s}}^* = \lambda\tilde{\mathbf{s}}^*$$

OPTIMAL FILTER: SOLUTION

FILTER (Normalized so $\mathbf{h}\mathbf{R}_{\boldsymbol{\eta}}\mathbf{h}^{*T} = 1$)

$$\mathbf{h} = \frac{1}{\sqrt{\mathbf{s}^{*T}\mathbf{R}_{\boldsymbol{\eta}}^{-1}\mathbf{s}}}\mathbf{R}_{\boldsymbol{\eta}}^{-1}\tilde{\mathbf{s}}^{*}$$

SIGNAL-NOISE

$$\mathsf{SNR}_{MAX} = \lambda_{MAX} = \mathbf{s}^{*T}\mathbf{R}_{\boldsymbol{\eta}}^{-1}\mathbf{s}$$

OPTIMAL FILTER: WHITE NOISE CASE $(\mathrm{R}_\eta = \sigma_o^2 \mathrm{I})$

FILTER

$$\mathbf{h} = \frac{1}{\sigma_o ||\mathbf{s}||} \tilde{\mathbf{s}}^*$$

SIGNAL-NOISE

$$\mathsf{SNR} = \frac{\|\mathbf{s}\|^2}{\sigma_o^2}$$

Insert Example 5.4 here.