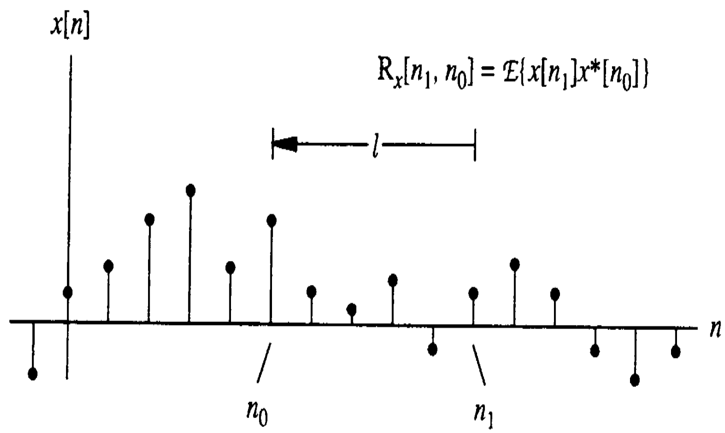
SECOND MOMENT CHARACTERIZATION OF RANDOM PROCESSES



MEAN

$$m_x[n] = \mathcal{E}\{x[n]\}$$

CORRELATION FUNCTION

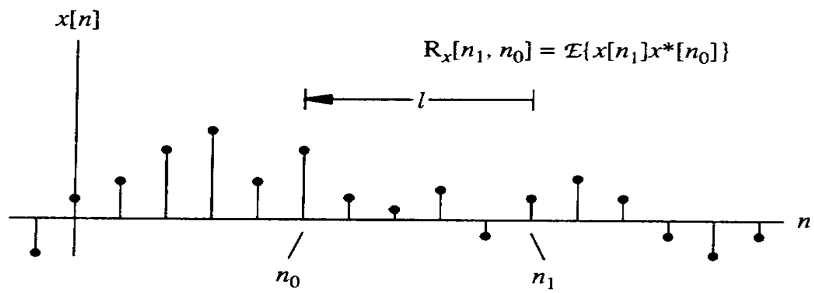
$$R_x[n_1, n_0] = \mathcal{E}\{x[n_1]x^*[n_0]\}$$

COVARIANCE FUNCTION

$$C_x[n_1, n_0] = \mathcal{E}\{(x[n_1] - m_x[n_1])(x[n_0] - m_x[n_0])^*\}$$

RELATION: $R_x[n_1, n_0] = C_x[n_1, n_0] + m_x[n_1]m_x^*[n_0]$

SECOND MOMENT CHARACTERIZATION (cont'd.)



For a stationary random process:

$$\mathrm{m}_x[n] = m_x \quad \text{(a constant)}$$

$$\mathrm{R}_x[n_1, n_0] = R_x[n_1 - n_0] = R_x[l] \qquad l \text{ is called the "lag"}$$

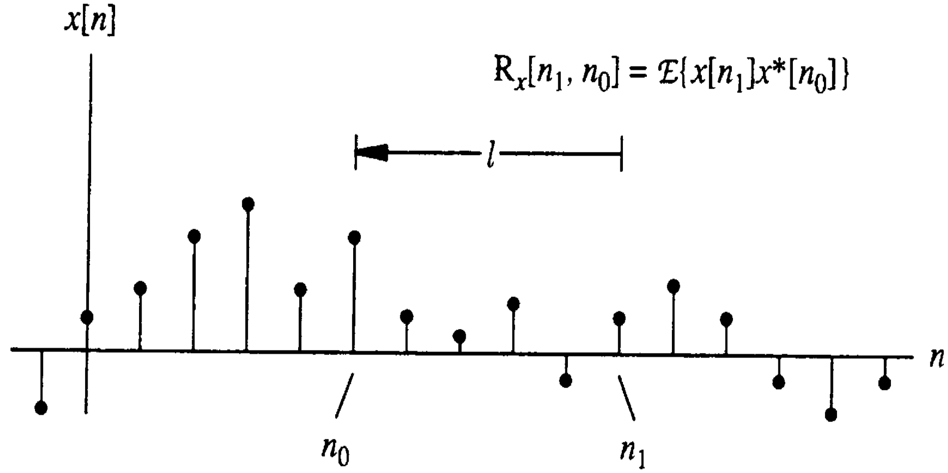
$$\mathrm{C}_x[n_1, n_0] = C_x[n_1 - n_0] = C_x[l]$$

- If a process is not stationary (in the strict sense) but satisfies the above conditions, it is called *wide sense stationary*.

CORRELATION AND COVARIANCE FUNCTIONS (STATIONARY RANDOM PROCESS)

CORRELATION FUNCTION

$$R_x[l] = \mathcal{E}\{x[n]x^*[n-l]\}$$



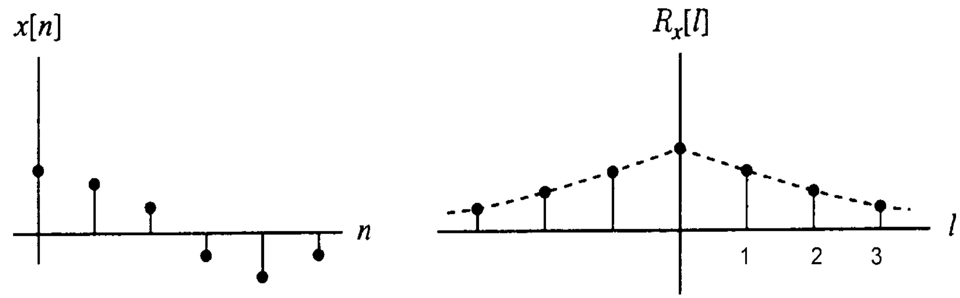
COVARIANCE FUNCTION

$$C_x[l] = \mathcal{E}\{(x[n]-m_x)(x[n-l]-m_x)^*\} \qquad \text{where} \quad m_x = \mathcal{E}\{x[n]\}$$

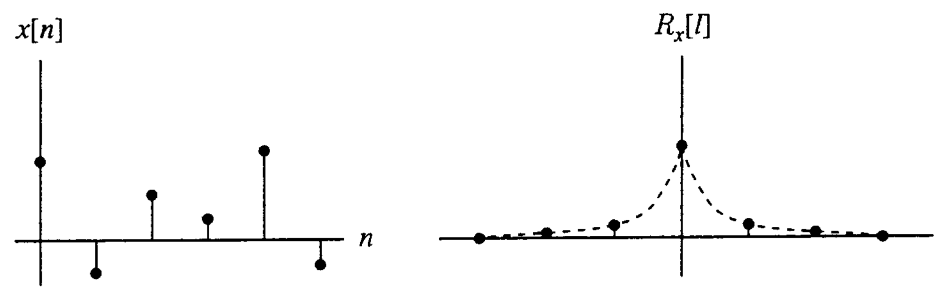
RELATION: $R_x[l] = C_x[l] + |m_x|^2$

EXAMPLES OF CORRELATION FUNCTIONS

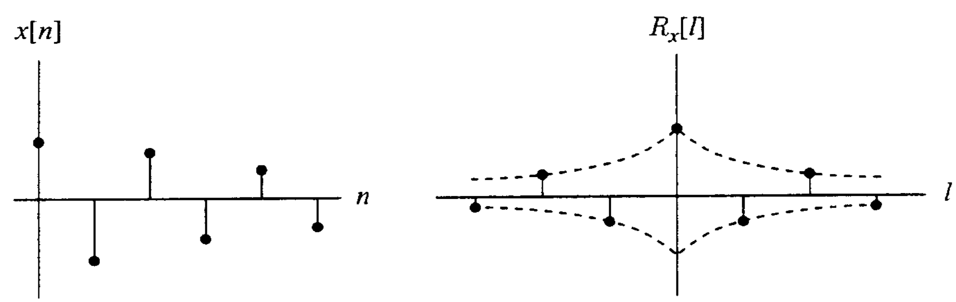
High correlation



Low correlation



Negative correlation



CORRELATION FUNCTION PROPERTIES

1. Conjugate symmetry

$$R_x[l] = R_x^*[-l]$$

2. Positive semidefinite property

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_0=-\infty}^{\infty} a^*[n_1] R_x[n_1 - n_0] a[n_0] \geq 0$$

for *any* sequence $a[n]$.

The second property *implies* that

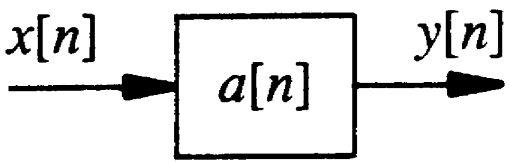
$$R_x[0] \geq |R_x[l]| \quad l \neq 0$$

PROOF OF PROPERTIES

Conjugate symmetry

$$R_x^*[-l] = \left(\mathcal{E}\left\{x[n]x^*[n+l]\right\}\right)^* = \mathcal{E}\left\{x[n+l]x^*[n]\right\} = R_x[l]$$

Positive semidefinite property



$$y[n] = \sum_{n_0=-\infty}^{\infty} a[n_0]x[n-n_0]$$

$$\begin{aligned}\mathcal{E}\left\{|y[n]|^2\right\} &= \sum_{n_1=-\infty}^{\infty}\sum_{n_0=-\infty}^{\infty} a^*[n_1]\mathcal{E}\left\{x[n-n_0]x^*[n-n_1]\right\}a[n_0] \\ &= \sum_{n_1=-\infty}^{\infty}\sum_{n_0=-\infty}^{\infty} a^*[n_1]R_x[n_1-n_0]a[n_0] \geq 0\end{aligned}$$

PROOF OF DERIVED PROPERTY $R_x[0] \geq |R_x[l]|$ (REAL CASE)

Start with:

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_0=-\infty}^{\infty} a^*[n_1] R_x[n_1 - n_0] a[n_0] \geq 0$$

and choose:

$$a[n] = \begin{cases} 1 & n = 0 \\ -1 & n = l \\ 0 & \text{otherwise} \end{cases}$$

The double sum becomes

$$\begin{aligned} & R_x[0-0] - R_x[0-l] - R_x[l-0] + R_x[l-l] \geq 0 \\ \Rightarrow \quad & R_x[0] \geq R_x[l] \end{aligned}$$

Then choose:

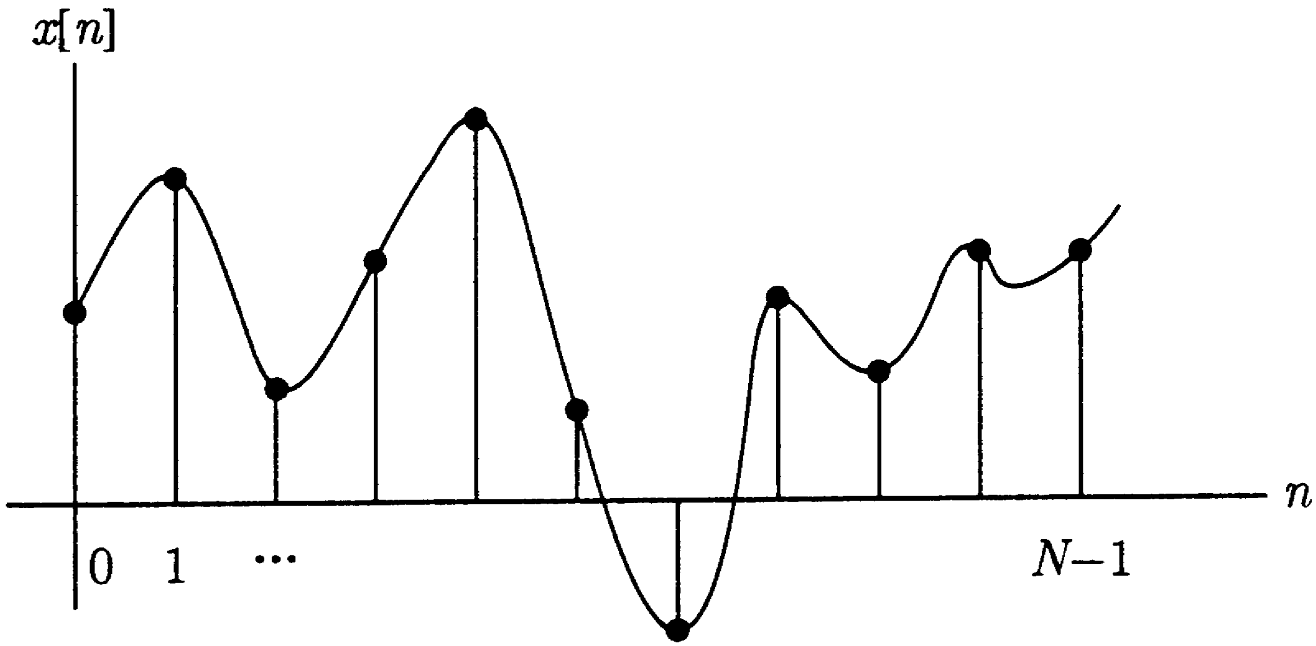
$$a[n] = \begin{cases} 1 & n = 0 \\ 1 & n = l \\ 0 & \text{otherwise} \end{cases}$$

to show that:

$$R_x[0] \geq -R_x[l]$$

Taken together: $R_x[0] \geq |R_x[l]|$

REPRESENTATION OF A RANDOM SIGNAL AS A RANDOM VECTOR



$$\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

CORRELATION MATRIX FOR A RANDOM PROCESS

$$\mathbf{R}_{\boldsymbol{x}} = \mathcal{E}\{\boldsymbol{x}\boldsymbol{x}^{*T}\} = \begin{bmatrix} \mathcal{E}\{|x[0]|^2\} & \mathcal{E}\{x[0]x^*[1]\} & \cdots & \mathcal{E}\{x[0]x^*[N-1]\} \\ \mathcal{E}\{x[1]x^*[0]\} & \mathcal{E}\{|x[1]|^2\} & \cdots & \mathcal{E}\{x[1]x^*[N-1]\} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{E}\{x[N-1]x^*[0]\} & \mathcal{E}\{x[N-1]x^*[1]\} & \cdots & \mathcal{E}\{|x[N-1]|^2\} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{R}_x[0,0] & \mathrm{R}_x[0,1] & \cdots & \mathrm{R}_x[0,N-1] \\ \mathrm{R}_x[1,0] & \mathrm{R}_x[1,1] & \cdots & \mathrm{R}_x[1,N-1] \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{R}_x[N-1,0] & \mathrm{R}_x[N-1,1] & \cdots & \mathrm{R}_x[N-1,N-1] \end{bmatrix}$$

CORRELATION MATRIX FOR A STATIONARY RANDOM PROCESS

$$
\mathbf{R}_{\boldsymbol{x}} = \begin{bmatrix} \mathrm{R}_x[0,0] & \mathrm{R}_x[0,1] & \cdots & \mathrm{R}_x[0,N-1] \\ \mathrm{R}_x[1,0] & \mathrm{R}_x[1,1] & \cdots & \mathrm{R}_x[1,N-1] \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{R}_x[N-1,0] & \mathrm{R}_x[N-1,1] & \cdots & \mathrm{R}_x[N-1,N-1] \end{bmatrix}
$$

$$
= \begin{bmatrix} R_x[0] & R_x[-1] & & & R_x[-N+1] \\ R_x[1] & R_x[0] & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & R_x[0] & R_x[-1] \\ R_x[N-1] & & & R_x[1] & R_x[0] \end{bmatrix}
$$

- $\mathbf{R}_{\boldsymbol{x}}$ is a *Toeplitz* matrix: elements along diagonals are equal.
- Covariance matrix $\mathbf{C}_{\boldsymbol{x}}$ is also Toeplitz with main diagonal elements $C_x[0] = \sigma_x^2$.

CROSS-CORRELATION AND -COVARIANCE

DEFINITIONS:

$$\mathsf{R}_{xy}[n_1, n_0] = \mathcal{E}\{x[n_1]y^*[n_0]\}$$

$$\mathsf{C}_{xy}[n_1, n_0] = \mathcal{E}\{(x[n_1] - \mathsf{m}_x[n_1])(y[n_0] - \mathsf{m}_y[n_0])^*\}$$

RELATION: $\mathsf{R}_{xy}[n_1, n_0] = \mathsf{C}_{xy}[n_1, n_0] + \mathsf{m}_x[n_1]\mathsf{m}_y^*[n_0]$

- $x[n]$ and $y[n]$ are *jointly stationary* (wide sense) if

    1. $x[n]$ and $y[n]$ are each stationary

    2. $\mathsf{R}_{xy}[n_1, n_0] = R_{xy}[n_1 - n_0]$

CROSS-CORRELATION AND -COVARIANCE (JOINTLY STATIONARY PROCESSES)

DEFINITIONS

$$R_{xy}[l] = \mathcal{E}\{x[n]y^*[n-l]\} \ ; \quad C_{xy}[l] = \mathcal{E}\{(x[n]-m_x)(y[n-l]-m_y)^*\}$$

RELATION

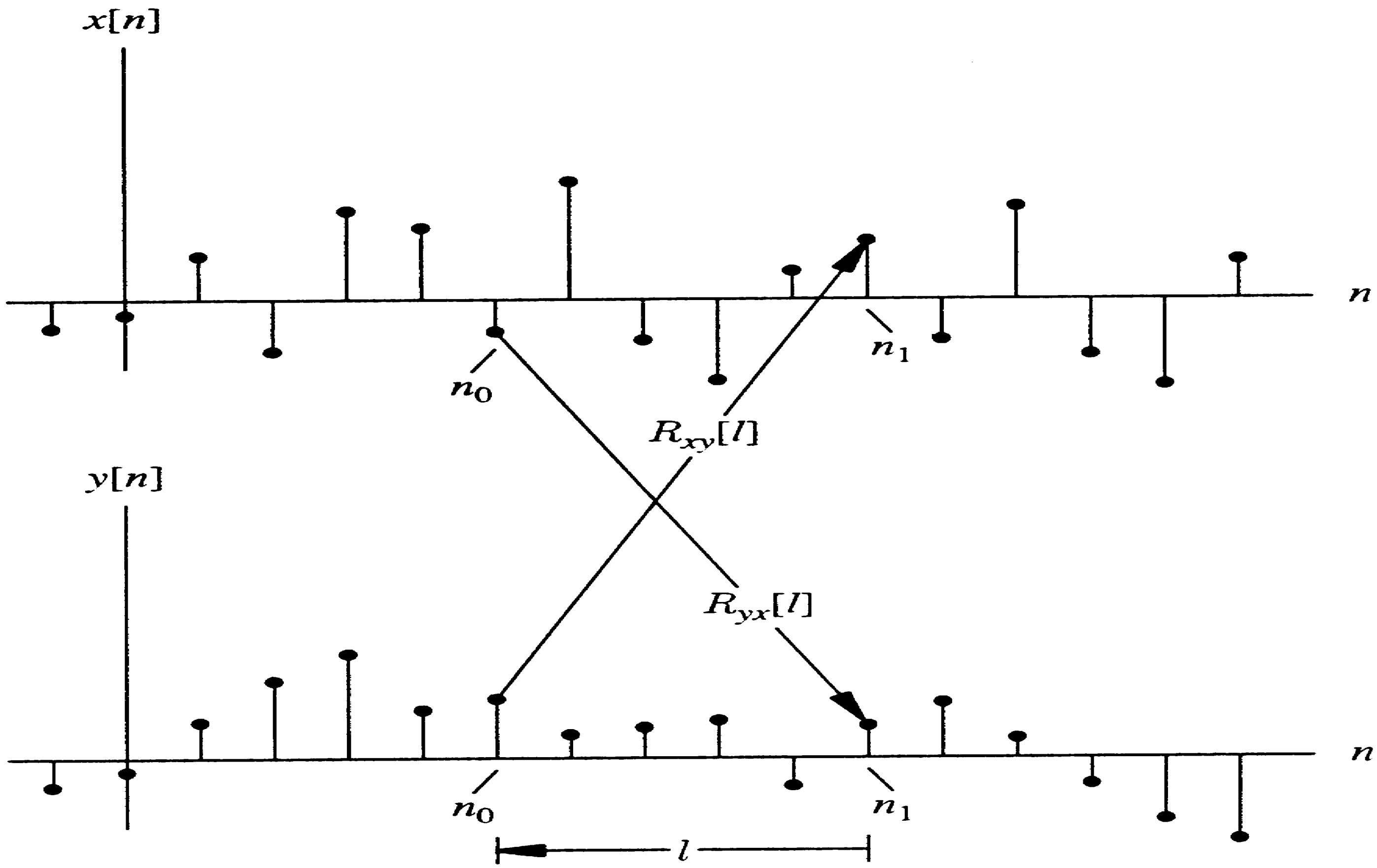
$$R_{xy}[l] = C_{xy}[l] + m_x m_y^*$$

"PROPERTIES"

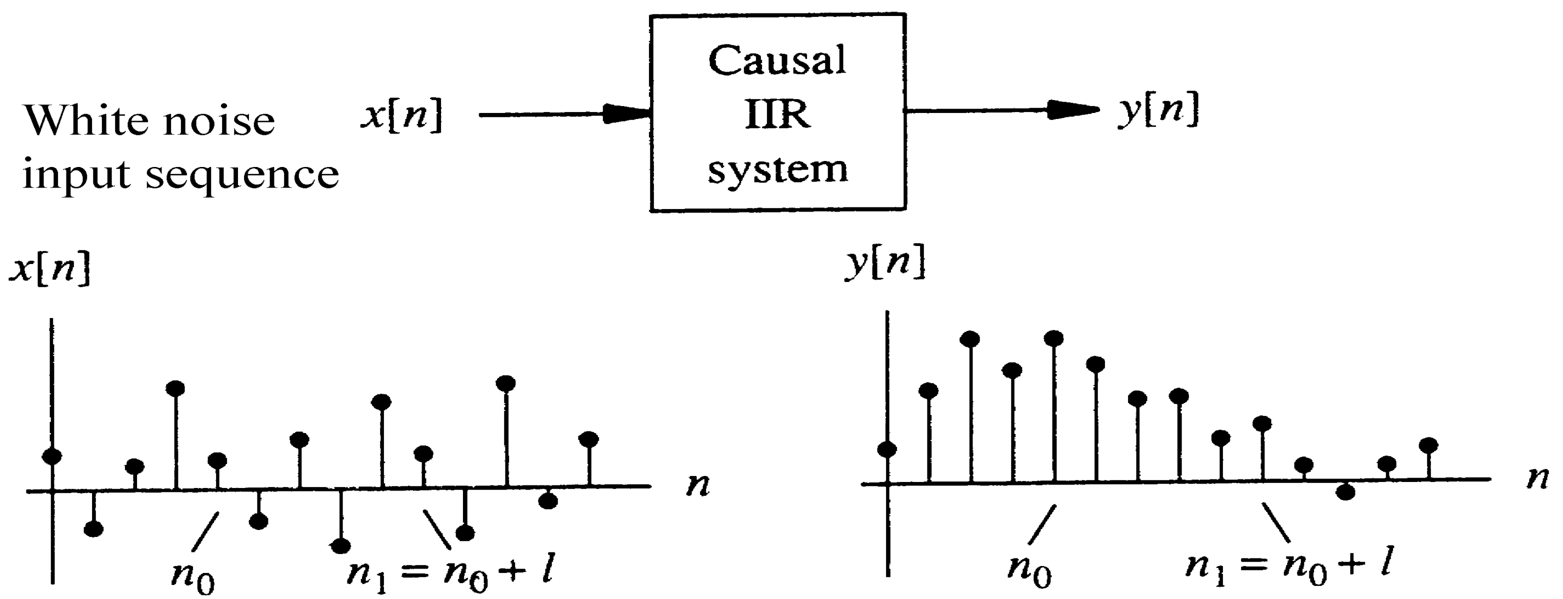
$$R_{xy}[l] = R_{yx}^*[-l] \ ; \qquad C_{xy}[l] = C_{yx}^*[-l]$$

These functions are usually *not* positive semidefinite.

CROSS-CORRELATION ILLUSTRATED

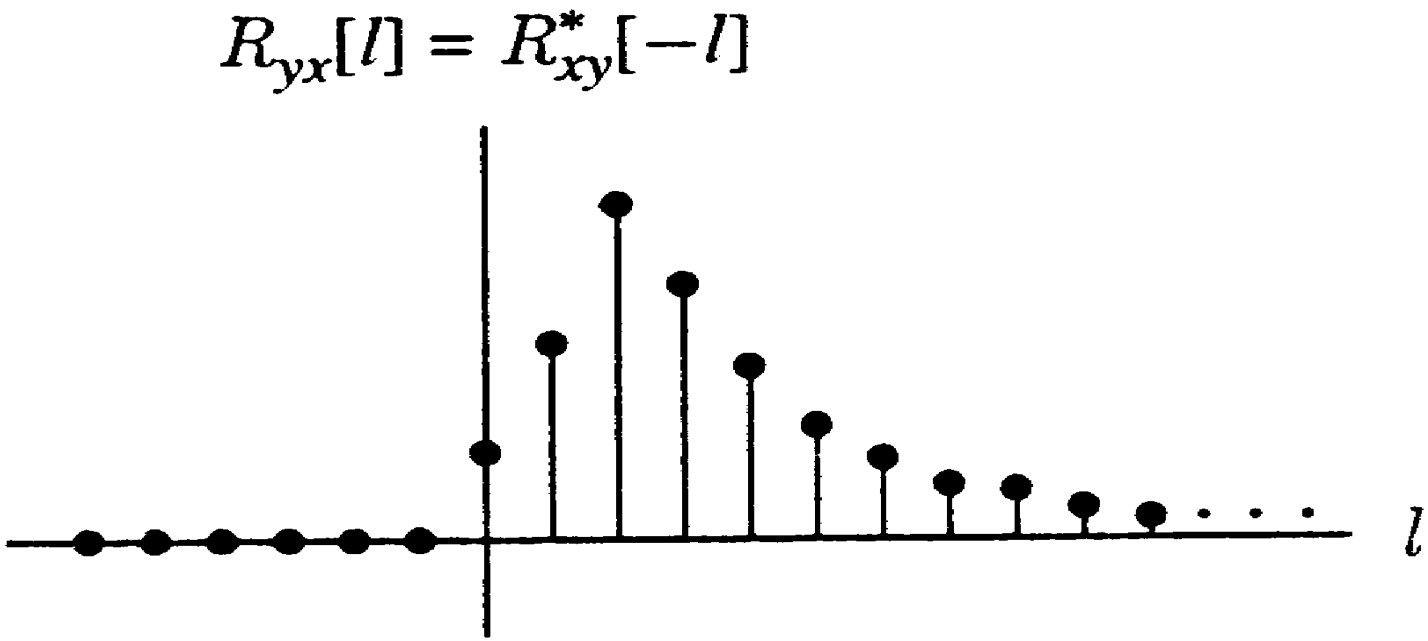


ASYMMETRY OF CROSS-CORRELATION



$$\mathcal{E}\{y[n_1]x^*[n_0]\} = R_{yx}[l] \neq 0$$
$$\mathcal{E}\{x[n_1]y^*[n_0]\} = R_{xy}[l] = 0$$

ASYMMETRY OF CROSS-CORRELATION (cont'd.)



BOUNDS ON CROSS-CORRELATION

ARITHMETIC MEAN

$$|R_{xy}[l]| \leq \tfrac{1}{2}\left(R_x[0] + R_y[0]\right)$$

GEOMETRIC MEAN

$$|R_{xy}[l]| \leq \left(R_x[0] R_y[0]\right)^{\frac{1}{2}}$$

- Analogous bounds hold for the covariance.

CROSS-CORRELATION AND -COVARIANCE MATRICES

$$\mathbf{R}_{\boldsymbol{xy}} = \mathcal{E}\left\{\boldsymbol{x}\boldsymbol{y}^{*T}\right\} \qquad \mathbf{C}_{\boldsymbol{xy}} = \mathcal{E}\left\{(\boldsymbol{x} - \mathbf{m}_{\boldsymbol{x}})(\boldsymbol{y} - \mathbf{m}_{\boldsymbol{y}})^{*T}\right\}$$

For stationary processes, these matrices are Toeplitz but not necessarily square.

<u>Example:</u>

$$\mathbf{R}_{\boldsymbol{xy}} = \begin{bmatrix} R_{xy}[0] & R_{xy}[-1] & R_{xy}[-2] & R_{xy}[-3] \\ R_{xy}[1] & R_{xy}[0] & R_{xy}[-1] & R_{xy}[-2] \\ R_{xy}[2] & R_{xy}[1] & R_{xy}[0] & R_{xy}[-1] \end{bmatrix}$$

POWER SPECTRAL DENSITY FUNCTION

DEFINITION AND INVERSE

$$S_x(e^{\jmath\omega}) = \sum_{l=-\infty}^{\infty} R_x[l] e^{-\jmath\omega l} \qquad R_x[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(e^{\jmath\omega}) e^{\jmath\omega l} d\omega$$

INTERPRETATION

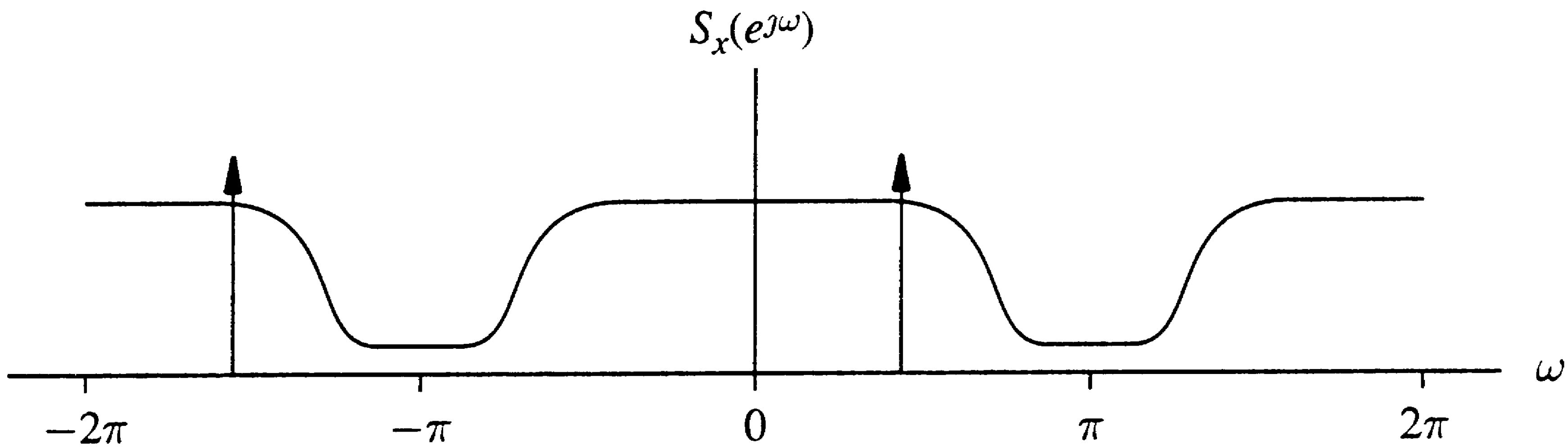
Since . . .

$$\text{Avg. Power} = \mathcal{E}\left\{|x[n]|^2\right\} = R_x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(e^{\jmath\omega}) d\omega$$

. . . $S_x(e^{\jmath\omega})$ must be power *density*.

POWER SPECTRAL DENSITY (cont'd.)

GENERAL FORM



$$S_x(e^{\jmath\omega}) = S'_x(e^{\jmath\omega}) + \sum_i 2\pi \mathsf{P}_i \delta_c(e^{\jmath\omega} - e^{\jmath\omega_i})$$

POWER SPECTRAL DENSITY PROPERTIES

1. $S_x(e^{j\omega})$ is <u>*real*</u>

2. $S_x(e^{j\omega}) \geq 0$

1 follows from the conjugate symmetry of $R_x[l]$.

2 follows from the positive semidefinite property of $R_x[l]$.

- $S_x(e^{j\omega})$ is also periodic, and is an *even* function of ω only if $R_x[l]$ is *real*.

PROOF OF PROPERTIES

1. $\underline{S_x(e^{j\omega}) \text{ is } \textit{real}}$

$$S_x^*(e^{j\omega}) = \left(\sum_{l=-\infty}^{\infty} R_x[l]e^{-j\omega l}\right)^* = \sum_{l=-\infty}^{\infty} R_x^*[l]e^{j\omega l}$$

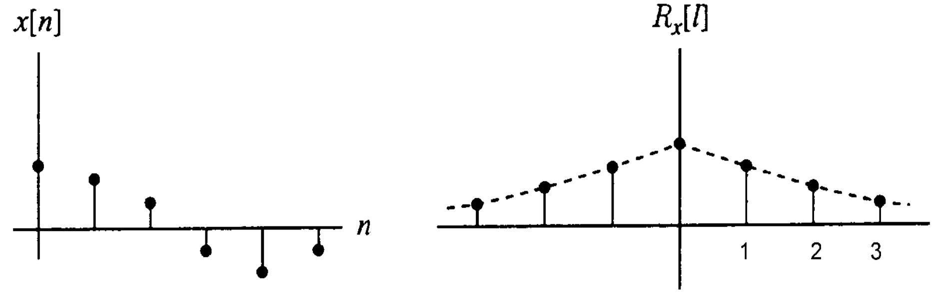
$$= \sum_{l=-\infty}^{\infty} R_x[-l]e^{j\omega l} = \sum_{k=-\infty}^{\infty} R_x[k]e^{-j\omega k} = S_x(e^{j\omega})$$

2. $\underline{S_x(e^{j\omega}) \geq 0}$

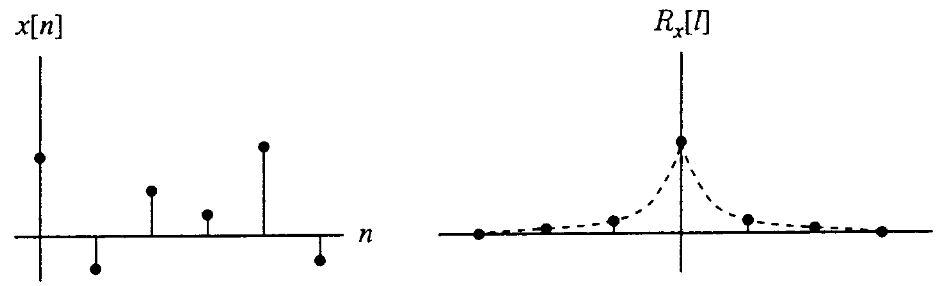
   Deferred to Chapter 5.

EXAMPLES OF CORRELATION (Review)

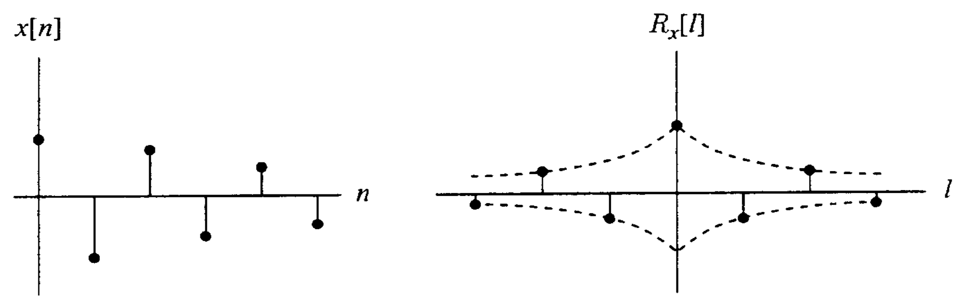
High correlation



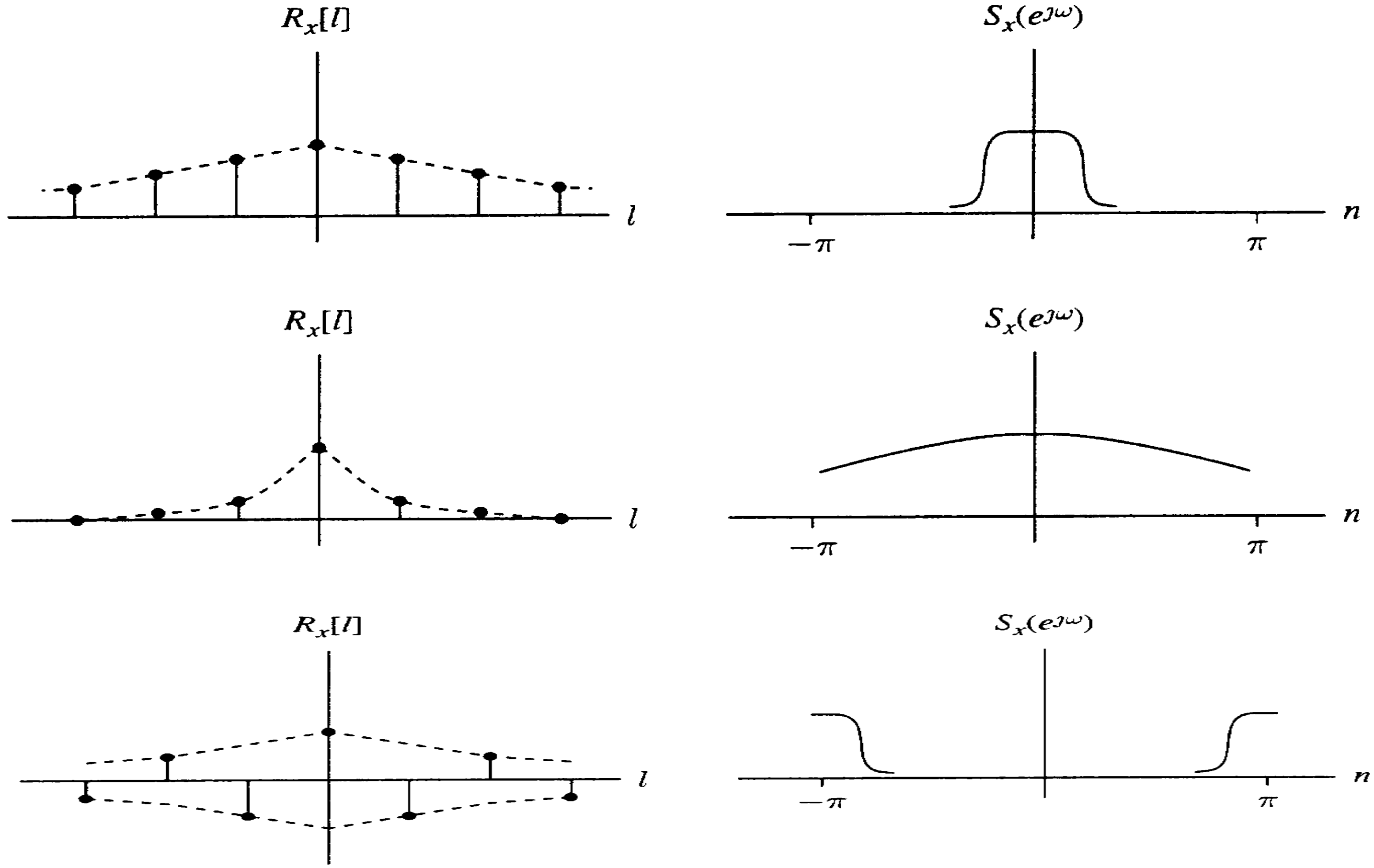
Low correlation



Negative correlation



EXAMPLES OF POWER SPECTRAL DENSITY



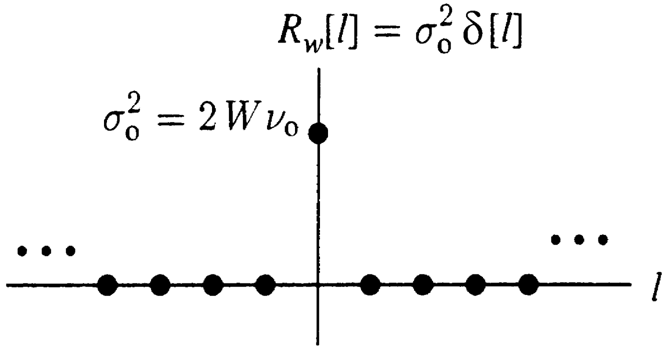
SOME IMPORTANT RANDOM PROCESSES

- White noise
- Signal with exponential correlation function

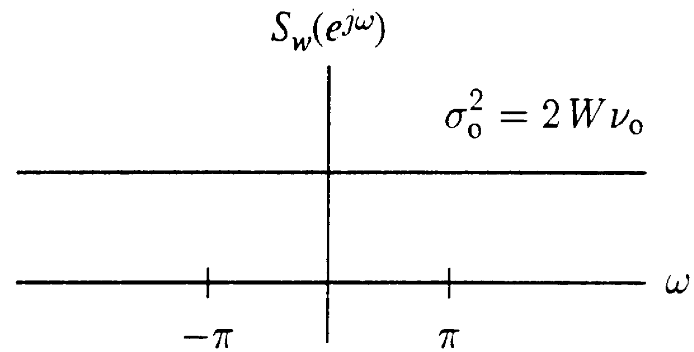
WHITE NOISE

- Any stationary random process, with zero mean, whose correlation function is an impulse is called “white noise.”

CORRELATION FUNCTION

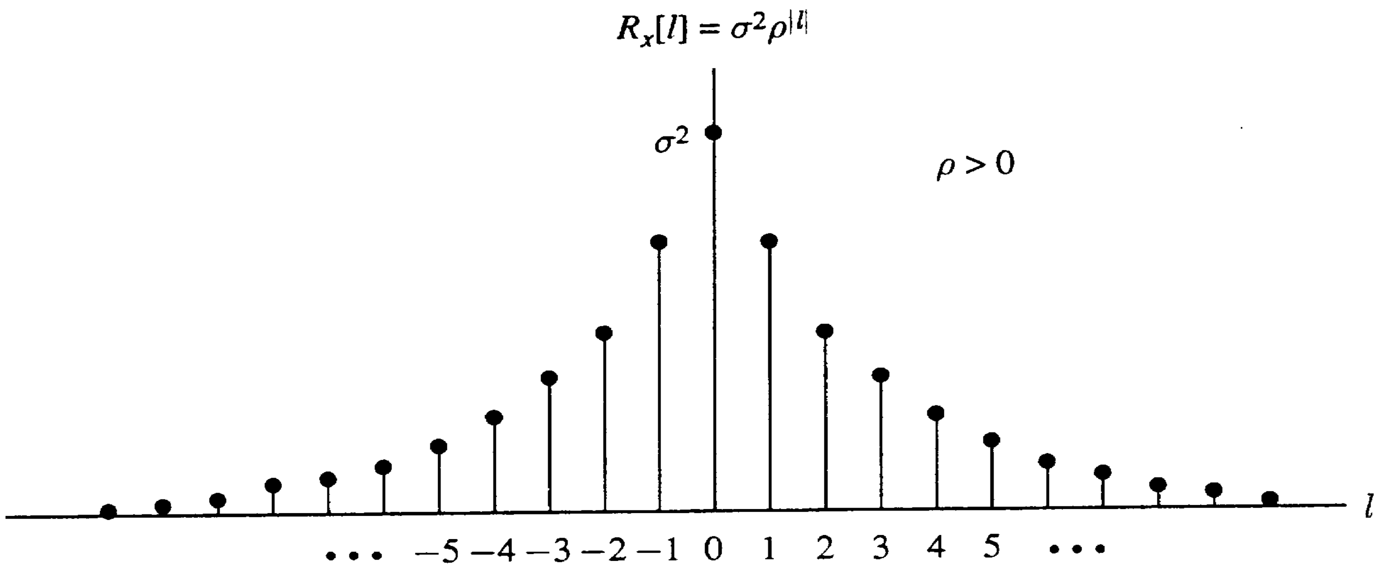


POWER SPECTRUM



SIGNAL WITH EXPONENTIAL CORRELATION

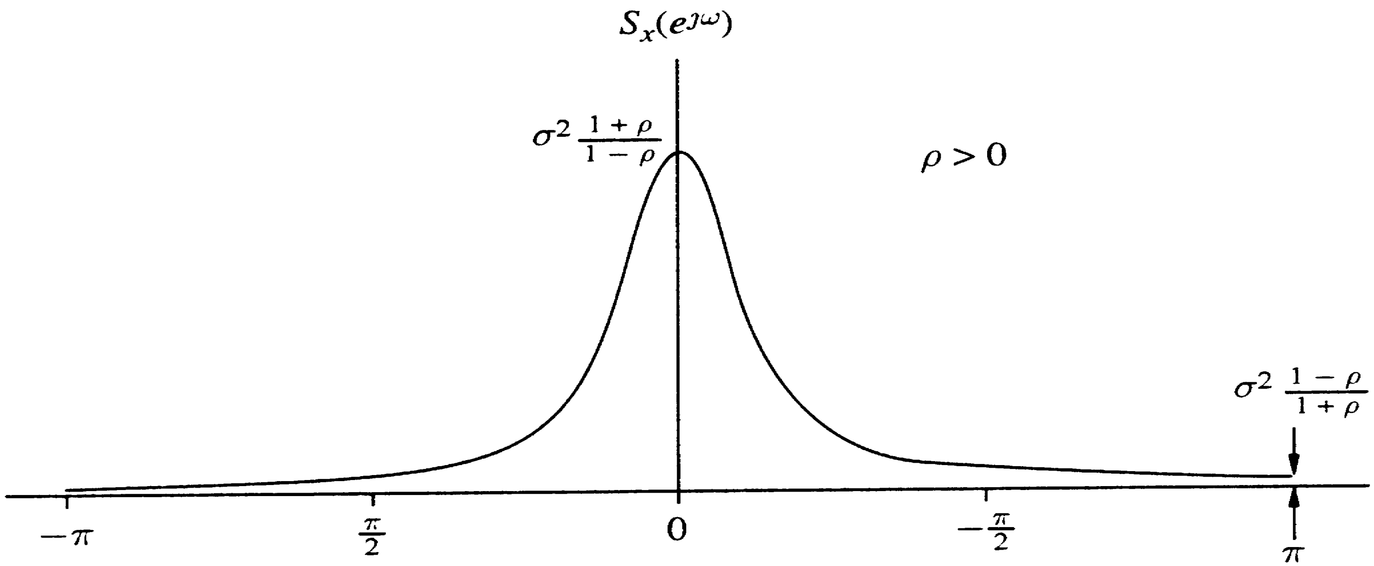
CORRELATION FUNCTION



$$R_x[l] = \sigma^2 \rho^{|l|} \quad -\infty < l < \infty \qquad (|\rho| < 1)$$

SIGNAL WITH EXPONENTIAL CORRELATION (cont'd.)

POWER SPECTRUM



$$S_x(e^{j\omega}) = \frac{\sigma^2(1-\rho^2)}{1+\rho^2-2\rho\cos\omega}$$

CROSS-POWER SPECTRAL DENSITY

DEFINITION AND INVERSE

$$S_{xy}(e^{\jmath\omega}) = \sum_{l=-\infty}^{\infty} R_{xy}[l]e^{-\jmath\omega l} \qquad R_{xy}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xy}(e^{\jmath\omega})e^{\jmath\omega l}d\omega$$

INTERPRETATION

Measures correlation between $x[n]$ and $y[n]$ *at frequency* ω.

"PROPERTIES"

Generally <u>*complex*</u> with $S_{xy}(e^{\jmath\omega}) = S^*_{yx}(e^{\jmath\omega})$.

NORMALIZED CROSS-SPECTRUM / COHERENCE

COHERENCE FUNCTION

$$\Gamma_{xy}(e^{\jmath\omega}) \stackrel{\text{def}}{=} \frac{S_{xy}(e^{\jmath\omega})}{\sqrt{S_x(e^{\jmath\omega})}\sqrt{S_y(e^{\jmath\omega})}}$$

MAGNITUDE SQUARED COHERENCE (MSC)

$$|\Gamma_{xy}(e^{\jmath\omega})|^2 = \frac{|S_{xy}(e^{\jmath\omega})|^2}{S_x(e^{\jmath\omega})S_y(e^{\jmath\omega})} \ ; \ 0 \leq |\Gamma_{xy}(e^{\jmath\omega})|^2 \leq 1$$

COMPLEX SPECTRAL DENSITY FUNCTION

DEFINITION

$$S_x(z) = \sum_{l=-\infty}^{\infty} R_x[l] z^{-l} \qquad (z \in \text{Region of convergence})$$

INVERSE

$$R_x[l] = \frac{1}{2\pi j} \oint_C S_x(z) z^{l-1} dz \qquad (C \in \text{Region of convergence})$$

SYMMETRY PROPERTY

$$S_x(z) = S_x^*(1/z^*) \qquad \Big(\text{for real processes: } S_x(z) = S_x(z^{-1})\Big)$$

COMPLEX SPECTRAL DENSITY (cont'd.)

GENERAL FORM

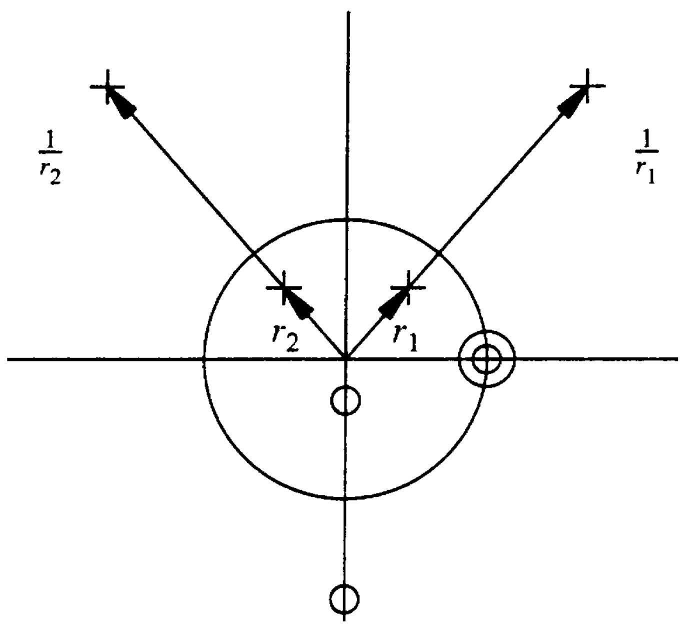
$$S_x(z) = S'_x(z) + \sum_i 2\pi \mathsf{P}_i \delta_c(z - e^{\jmath\omega_i})$$

RATIONAL FORMS

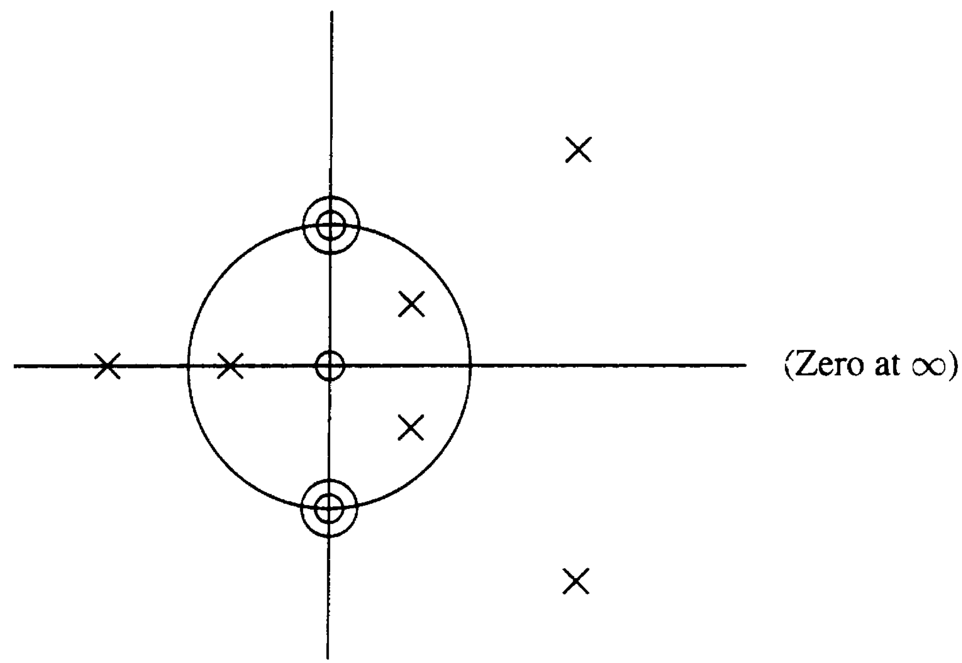
- $S'_x(z) = \frac{N(z)}{D(z)}$ example: $S'_x(z) = \frac{0.6z - 2 + 0.6z^{-1}}{z - 2.5 + z^{-1}}$

- Poles and zeros occur in *conjugate reciprocal* locations: z_o and $1/z_\mathsf{o}^*$

EXAMPLES OF POLE-ZERO LOCATIONS FOR A RATIONAL COMPLEX SPECTRAL DENSITY



Complex process



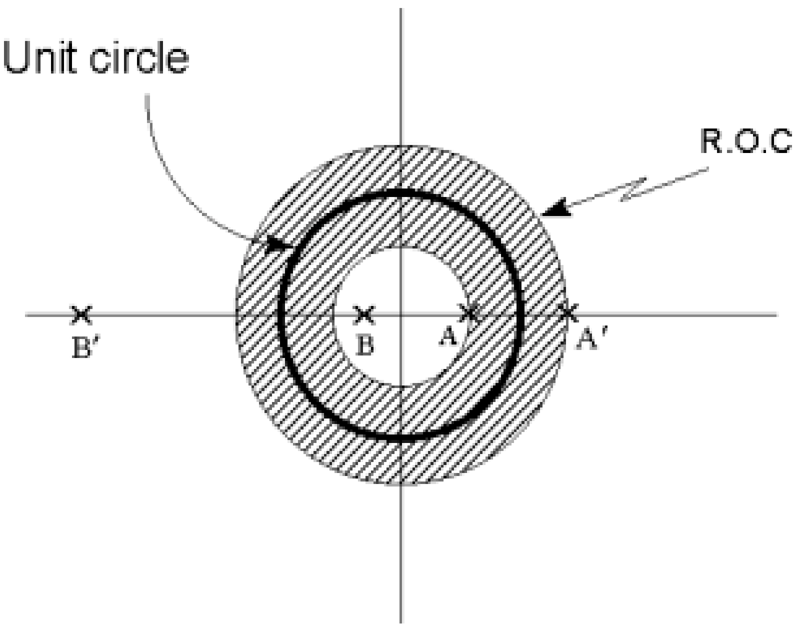
Real process

REGION OF CONVERGENCE OF $S_x(z)$

Since $S_x(z) = S_x^*\left(\frac{1}{z^*}\right)$ then if $S_x(z)$ converges for $|z| > r_R$ then $S_x(z)$ also converges for $\frac{1}{|z^*|} > r_R$ or $|z| < \frac{1}{r_R}$

Therefore, the region of convergence is always of the form

$$r_R < |z| < \frac{1}{r_R}$$



EXPONENTIAL CORRELATION FUNCTION AND COMPLEX SPECTRAL DENSITY

	Correlation Function	Complex Spectral Density
Real Case	$R_x[l] = \sigma^2 \rho^{\lvert l \rvert}$	$S_x(x) = \dfrac{\sigma^2(1-\rho^2)}{-\rho z + (1+\rho^2) - \rho z^{-1}}$
Complex Case	$R_x[l] = \begin{cases} \sigma^2 \rho^l & l \geq 0 \\ \sigma^2 (\rho^*)^{-l} & l < 0 \end{cases}$	$S_x(x) = \dfrac{\sigma^2(1-\lvert\rho\rvert^2)}{-\rho^* z + (1+\lvert\rho\rvert^2) - \rho z^{-1}}$

- Also see various other forms (Eq. 4.61 of text).

COMPLEX CROSS-SPECTRAL DENSITY

DEFINITION

$$S_{xy}(z) = \sum_{l=-\infty}^{\infty} R_{xy}[l] z^{-l} \qquad (z \in \text{Region of convergence})$$

INVERSE

$$R_{xy}[l] = \frac{1}{2\pi j} \oint_C S_{xy}(z) z^{l-1} dz \qquad (C \in \text{Region of convergence})$$

“PROPERTY”

$$S_{xy}(z) = S_{yx}^{*}(1/z^{*}) \qquad \Big(\text{for real processes: } S_{xy}(z) = S_{yx}(z^{-1})\Big)$$

SUMMARY OF 2^{nd} MOMENT FUNCTIONS AND PROPERTIES: AUTOCORRELATION

Function and Definition	Properties
$R_x[l] = \mathcal{E}\{x[n]x^*[n-l]\}$	$R_x[l] = R_x^*[-l]$ $\sum_{n_1=-\infty}^{\infty}\sum_{n_0=-\infty}^{\infty} a^*[n_1]R_x[n_1-n_0]a[n_0] \geq 0$
$S_x(e^{\jmath\omega}) = \sum_{l=-\infty}^{\infty} R_x[l]e^{-\jmath\omega l}$	$S_x(e^{\jmath\omega})$ is real. $S_x(e^{\jmath\omega}) \geq 0$
$S_x(z) = \sum_{l=-\infty}^{\infty} R_x[l]z^{-l}$	$S_x(z) = S_x^*(1/z^*)$

SUMMARY OF 2^{nd} MOMENT FUNCTIONS AND PROPERTIES: CROSS-CORRELATION

Function and Definition	Relations
$R_{xy}[l] = \mathcal{E}\{x[n]y^*[n-l]\}$	$R_{xy}[l] = R^*_{yx}[-l]$
$S_{xy}(e^{j\omega}) = \sum_{l=-\infty}^{\infty} R_{xy}[l]e^{-j\omega l}$	$S_{xy}(e^{j\omega}) = S^*_{yx}(e^{j\omega})$
$S_{xy}(z) = \sum_{l=-\infty}^{\infty} R_{xy}[l]z^{-l}$	$S_{xy}(z) = S^*_{yx}(1/z^*)$

SECOND MOMENT ANALYSIS OF PERIODIC RANDOM PROCESSES

DENSITY FUNCTION

$$f_{x[n_0]x[n_1]\cdots x[n_L]} = f_{x[n_0+k_0P]x[n_1+k_1P]\cdots x[n_L+k_LP]}$$

MEAN

$$\mathcal{E}\{x[n]\} = \mathcal{E}\{x[n+kP]\} \qquad \Rightarrow \quad m_x[n] = m_x[n+kP]$$

CORRELATION

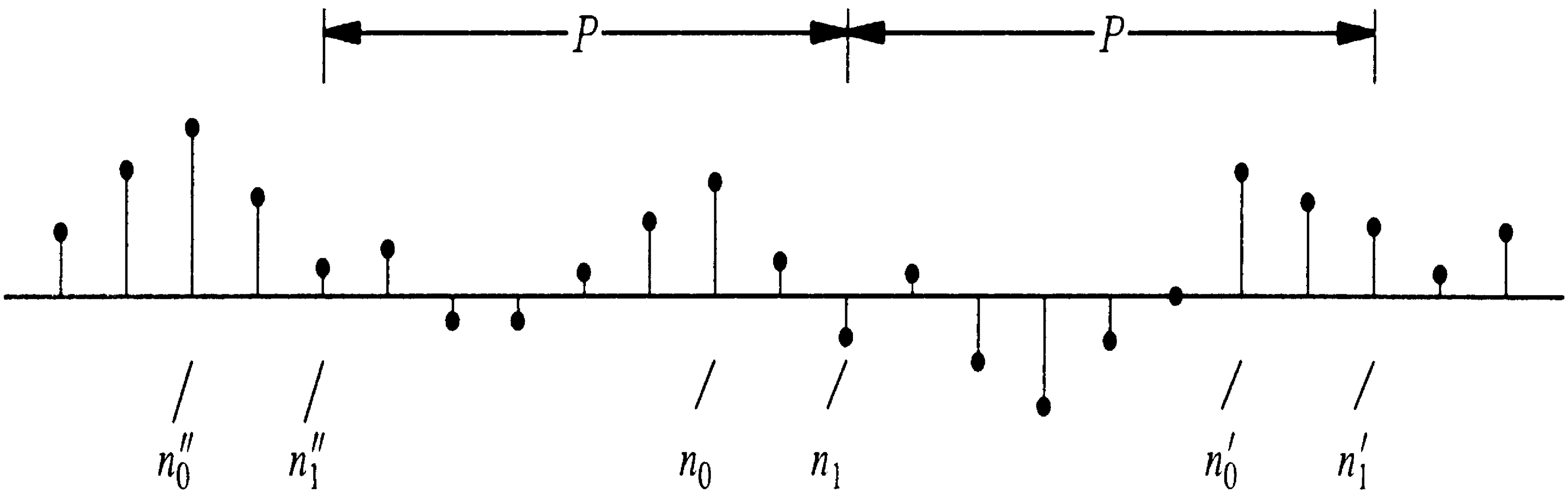
$$\begin{aligned} \mathcal{E}\{x[n_1]x^*[n_0]\} &= \mathcal{E}\{x[n_1+k_1P]x^*[n_0+k_0P]\} \\ \Rightarrow \qquad \mathsf{R}_x[n_1, n_0] &= \mathsf{R}_x[n_1+k_1P, n_0+k_0P] \end{aligned}$$

- Similarly $\mathsf{C}_x[n_1, n_0] = \mathsf{C}_x[n_1+k_1P, n_0+k_0P]$

PERIODIC RANDOM PROCESS

POINTS OF IDENTICAL CORRELATION

$$\mathsf{R}_x[n_1, n_0] = \mathsf{R}_x[n'_1, n_0] = \mathsf{R}_x[n''_1, n'_0] \;\; = \text{etc.}$$



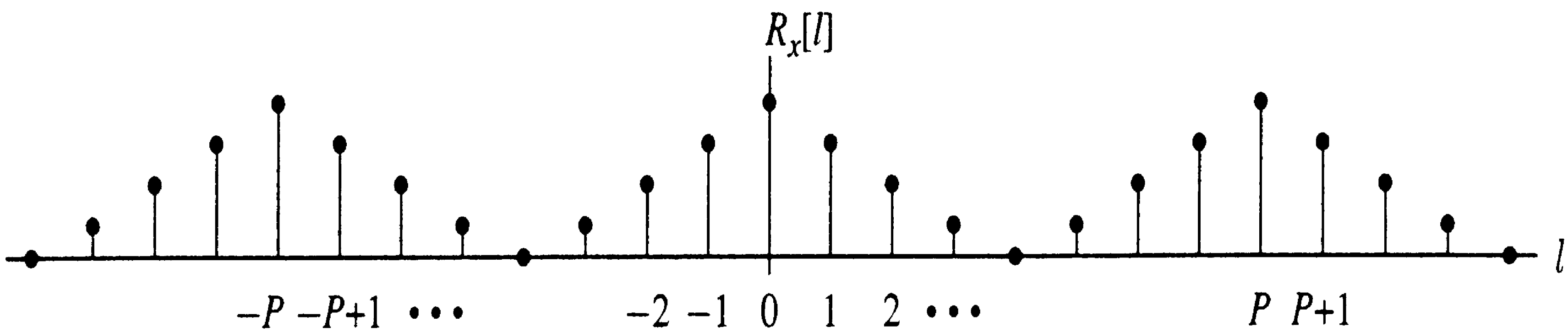
PERIODIC PROCESS: STATIONARY CASE

$$\mathsf{R}_x[n_1, n_0] = \mathsf{R}_x[n_1 + k_1 P, n_0 + k_0 P]$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

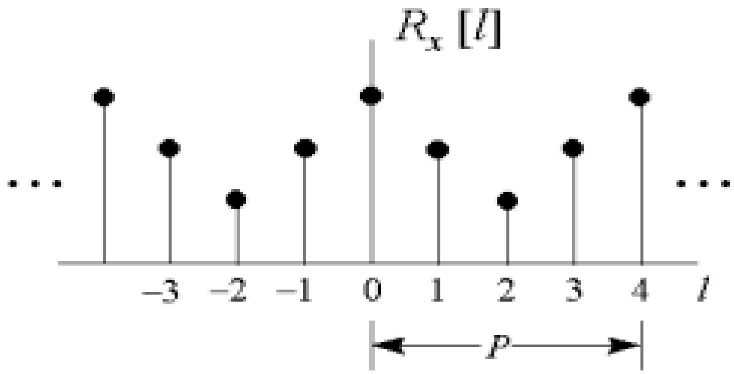
$$R_x[\underbrace{n_1 - n_0}_{l}] = R_x[n_1 + k_1 P - (n_0 + k_0 P)] = R_x[\underbrace{n_1 - n_0}_{l} + \underbrace{(k_1 - k_0)}_{k} P]$$

Therefore . . . $R_x[l] = R_x[l + kP]$



- Similarly $C_x[l] = C_x[l + kP]$

CORRELATION MATRIX FOR A PERIODIC RANDOM PROCESS



$R_x[1] = R_x[-3]$; $R_x[2] = R_x[-2]$; etc.

$$\mathbf{R}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} R_x[0] & R_x[-1] & R_x[-2] & R_x[-3] \\ R_x[1] & R_x[0] & R_x[-1] & R_x[-2] \\ R_x[2] & R_x[1] & R_x[0] & R_x[-1] \\ R_x[3] & R_x[2] & R_x[1] & R_x[0] \end{bmatrix}}_{\text{General form}} = \underbrace{\begin{bmatrix} R_x[0] & R_x[-1] & R_x[-2] & R_x[-3] \\ R_x[-3] & R_x[0] & R_x[-1] & R_x[-2] \\ R_x[-2] & R_x[-3] & R_x[0] & R_x[-1] \\ R_x[-1] & R_x[-2] & R_x[-3] & R_x[0] \end{bmatrix}}_{\text{Circulant matrix}}$$

EIGENVECTORS AND EIGENVALUES FOR A PERIODIC RANDOM PROCESS

The Discrete Fourier Transform $\mathsf{S}_x[k] = \sum_{l=0}^{3} R_x[l] W^{kl} \qquad \left(W = e^{-\jmath \frac{2\pi}{N}} \right)$

leads to the eigenvalue equation:

$$\begin{array}{l} \text{DFT} \quad \longrightarrow \\ \left.\begin{array}{l}\text{DFTs of} \\ \text{rotated} \\ \text{sequences}\end{array}\right\{ \end{array} \begin{bmatrix} R_x[0] & R_x[1] & R_x[2] & R_x[3] \\ R_x[3] & R_x[0] & R_x[1] & R_x[2] \\ R_x[2] & R_x[3] & R_x[0] & R_x[1] \\ R_x[1] & R_x[2] & R_x[3] & R_x[0] \end{bmatrix} \begin{bmatrix} 1 \\ W^k \\ W^{2k} \\ W^{3k} \end{bmatrix} = \mathsf{S}_x[k] \begin{bmatrix} 1 \\ W^k \\ W^{2k} \\ W^{3k} \end{bmatrix}$$

- Eigenvalues $\mathsf{S}_x[k]$ are *samples* of the power spectrum.

GENERAL CLASS OF PERIODIC PROCESSES

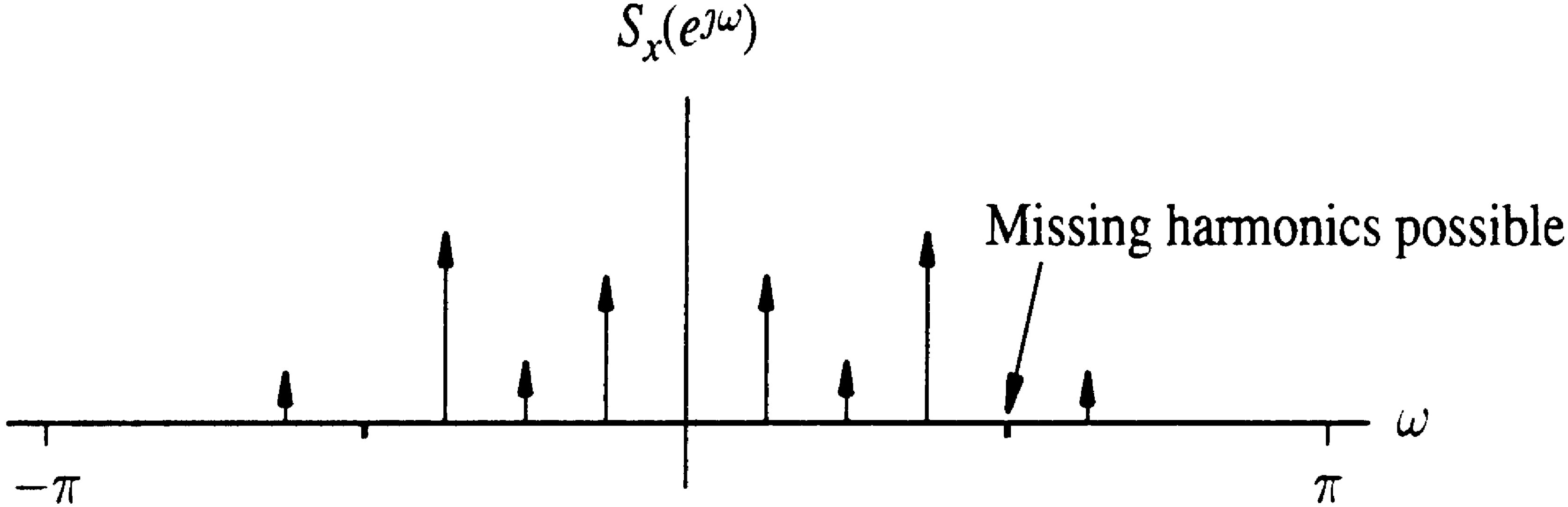
Random process[1]	$x[n] = \sum_i A_i e^{\jmath\omega_i n}$; $\mathcal{E}\left\{A_i A_j^*\right\} = 0, \quad i \neq j$
Correlation function	$R_x[l] = \sum_i \mathsf{P}_i e^{\jmath\omega_i l}$; $\mathsf{P}_i = \mathcal{E}\left\{\lvert A_i\rvert^2\right\} = \mathrm{Var}[A_i]$
Spectral density	$S_x(e^{\jmath\omega}) = 2\pi \sum_i \mathsf{P}_i \delta_c\left(e^{\jmath\omega} - e^{\jmath\omega_i}\right)$

[1] $A_i = |A_i| e^{\jmath\phi_i}$ with ϕ_i uniform.

- Model includes $x[n] = Ae^{\jmath\omega_0 n} + A^* e^{-\jmath\omega_0 n} = 2|A| \cos(\omega_0 n + \phi)$

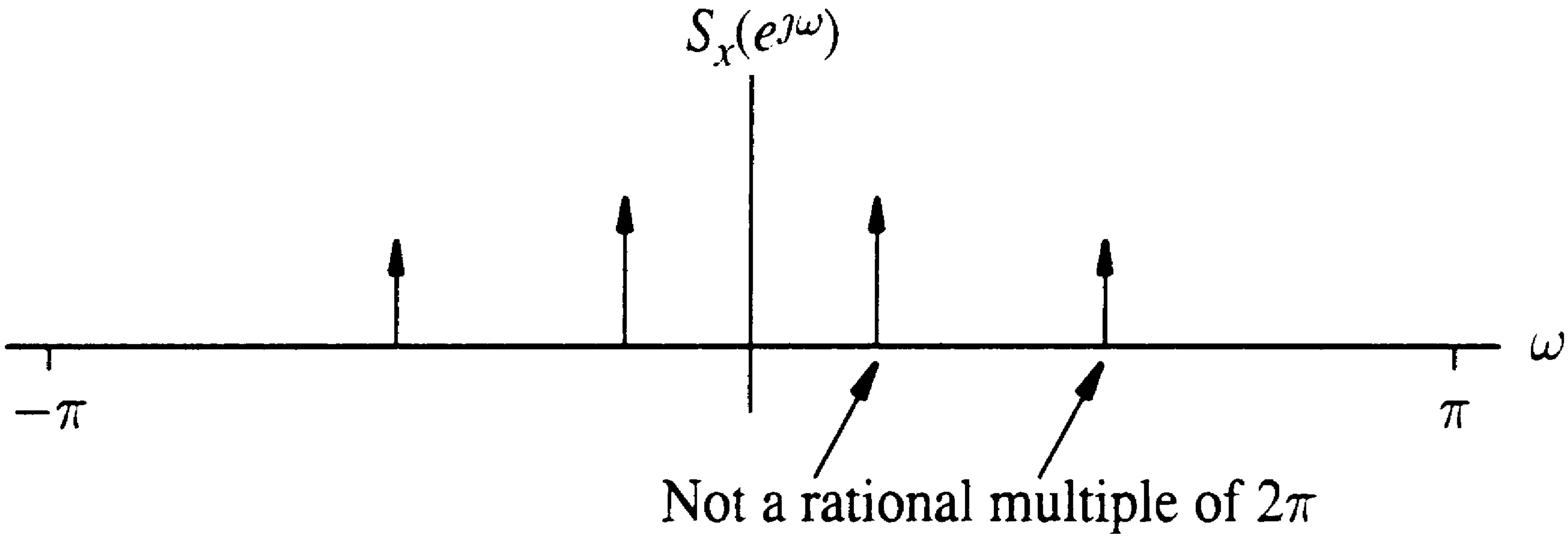
SPECTRA OF PERIODIC PROCESS

PERIODIC PROCESS

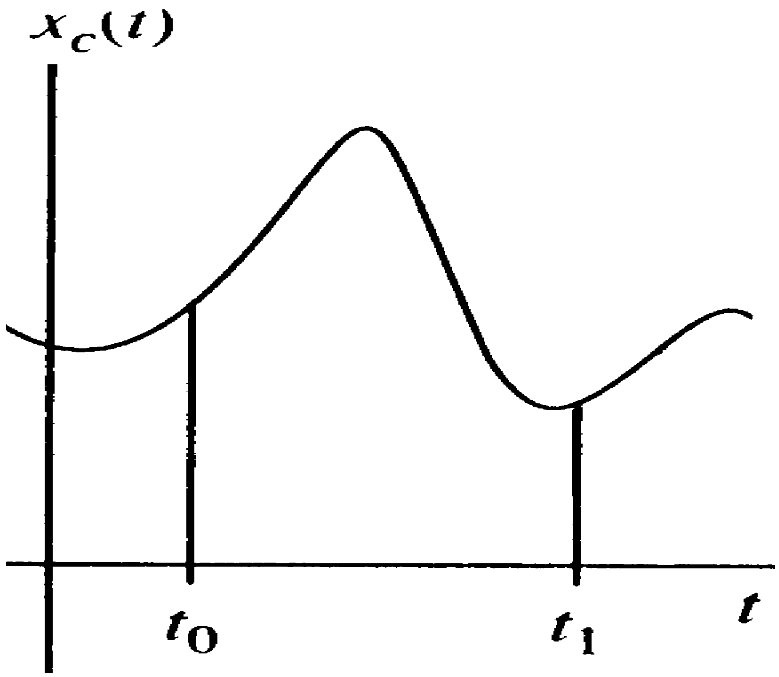


SPECTRA OF PERIODIC PROCESS (cont'd.)

ALMOST PERIODIC PROCESS



SECOND MOMENT CHARACTERIZATION OF CONTINUOUS RANDOM PROCESSES



MEAN

$$m_{x_c}^c(t) = \mathcal{E}\{x_c(t)\}$$

CORRELATION

$$\mathsf{R}_{x_c}^c(t_1, t_0) = \mathcal{E}\{x_c(t_1)x_c^*(t_0)\}$$

COVARIANCE

$$\mathsf{C}_{x_c}^c(t_1, t_0) = \mathcal{E}\left\{\Big(x_c(t_1) - m_{x_c}^c(t_1)\Big)\ \Big(x_c(t_0) - m_{x_c}^c(t_0)\Big)^*\right\}$$

SECOND MOMENT CHARACTERIZATION FOR A STATIONARY RANDOM PROCESS

MEAN

$$m_{x_c}^c = \mathcal{E}\{x_c(t)\}$$

CORRELATION

$$R_{x_c}^c(\tau) = \mathcal{E}\{x_c(t)x_c^*(t-\tau)\}$$

COVARIANCE

$$C_{x_c}^c(\tau) = \mathcal{E}\left\{(x_c(t) - m_{x_c}^c)(x_c(t-\tau) - m_{x_c}^c)^*\right\} = R_{x_c}^c(\tau) - |m_{x_c}^c|^2$$

PROPERTIES OF THE CONTINUOUS CORRELATION FUNCTION

1. Conjugate symmetry

$$R^c_{x_c}(\tau) = R^{c*}_{x_c}(-\tau)$$

2. Positive semidefinite property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_c^*(t_1)R_x^c(t_1 - t_0)a_c(t_0)dt_1 dt_0 \geq 0$$

for any function $a_c(t)$

The second property implies that

$$R_x(0) \geq |R_x(\tau)| \qquad \tau \neq 0$$

CONTINUOUS-DISCRETE RELATIONS

Let $x[n] \stackrel{\text{def}}{=} x_c(nT)$ (samples of the continuous process)

then

$$\begin{array}{rcl} \mathcal{E}\{x[n]x^*[n-l]\} & = & \mathcal{E}\{x_c(nT)x_c^*((n-l)T)\} \\ \searrow & & \swarrow \\ \text{Therefore}\ldots \quad R_x[l] & = & R_{x_c}^c(lT) \end{array}$$

- The discrete correlation function is comprised of *samples* of the continuous correlation function taken at the sampling interval T.

POWER SPECTRAL DENSITY FUNCTION

DEFINITION AND INVERSE

$$S^c_{x_c}(f) = \int_{-\infty}^{\infty} R^c_{x_c}(\tau) e^{-\jmath 2\pi f \tau} d\tau \quad ; \quad R^c_{x_c}(\tau) = \int_{-\infty}^{\infty} S^c_{x_c}(\tau) e^{\jmath 2\pi f \tau} df$$

PROPERTIES

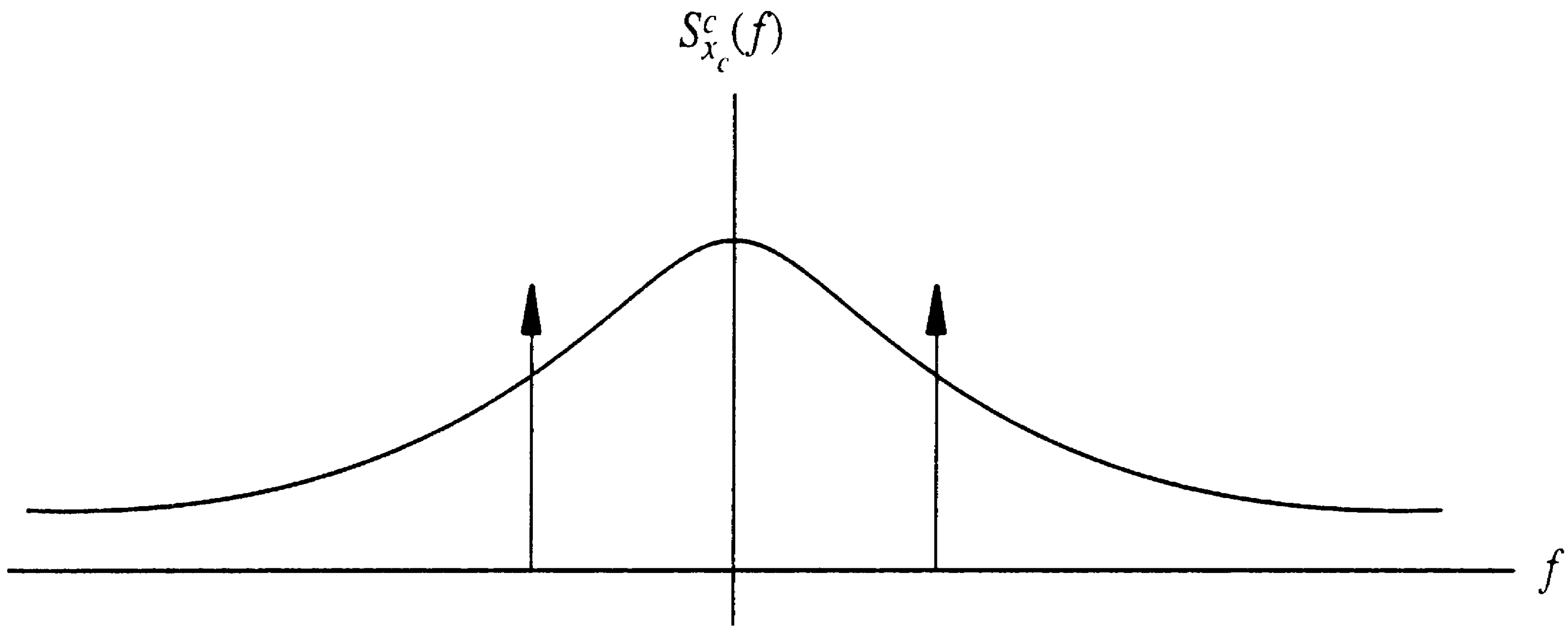
1. $S^c_{x_c}$ is *real* (and even if R_{x_c} is real).

2. $S^c_{x_c}$ is everywhere *non-negative*.

Note: $S^c_{x_c}(f)$ extends over $-\infty < f < \infty$; it is *not* periodic.

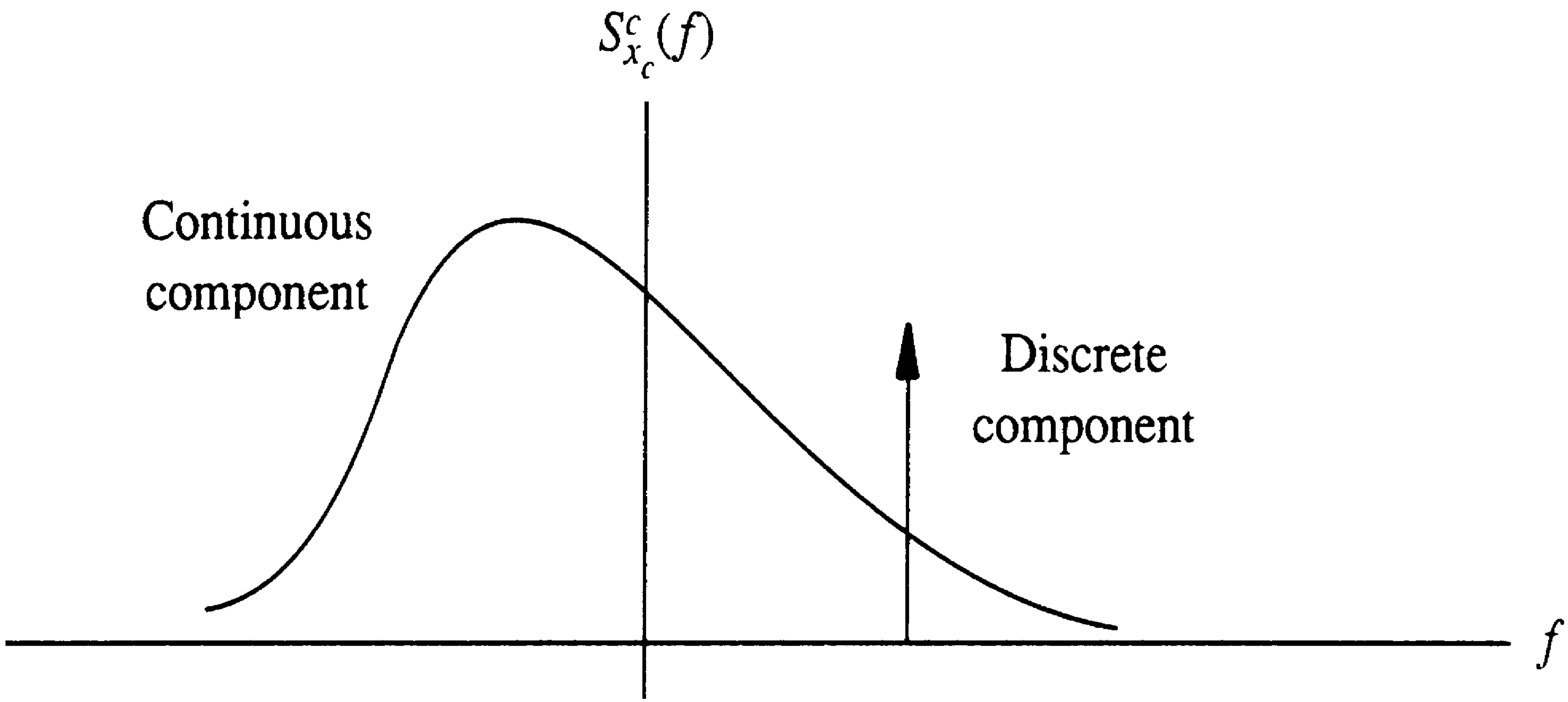
TYPICAL POWER DENSITY SPECTRUM

REAL RANDOM PROCESS



TYPICAL POWER DENSITY SPECTRUM

COMPLEX RANDOM PROCESS



SAMPLING AND RECONSTRUCTION

Theorem: If $S^c_{x_c}(f)$ is *bandlimited*, i.e., $S^c_{x_c}(f) = 0$; $|f| > W$ then $x_c(t)$ is reconstructable from samples $x_c(nT)$ taken at the Nyquist interval $T = 1/2W$ in the following sense:

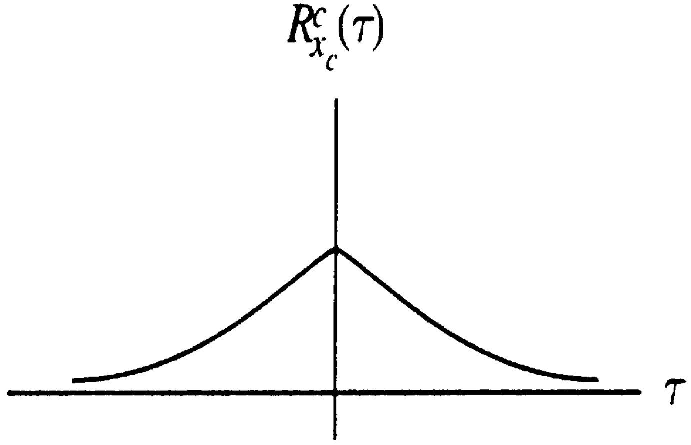
Let $$\breve{x}_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\text{sinc}\,(2\pi Wt - n\pi) \qquad \left(\text{sinc}\ (\xi) = \frac{\sin\xi}{\xi}\right)$$

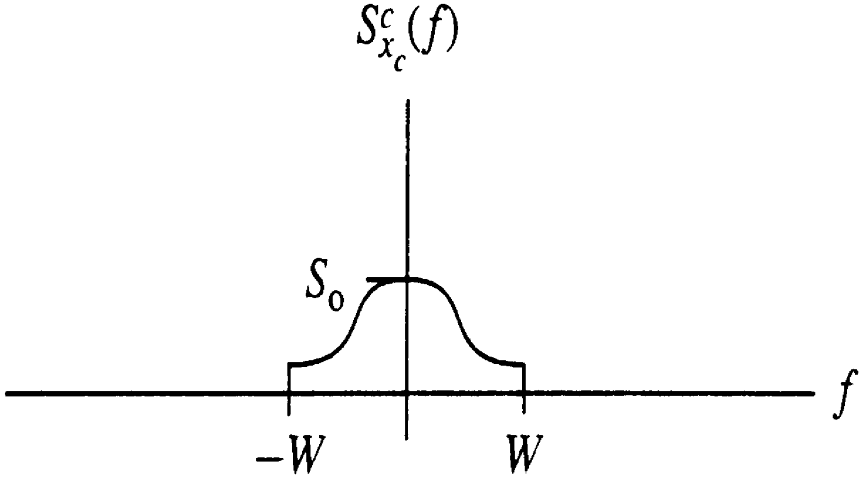
then $$\mathcal{E}\left\{|x_c(t) - \breve{x}_c(t)|^2\right\} = 0$$

This condition is written as $\breve{x}_c(t) \doteq x_c(t)$ (“almost everywhere”)

CONTINUOUS AND DISCRETE SPECTRA

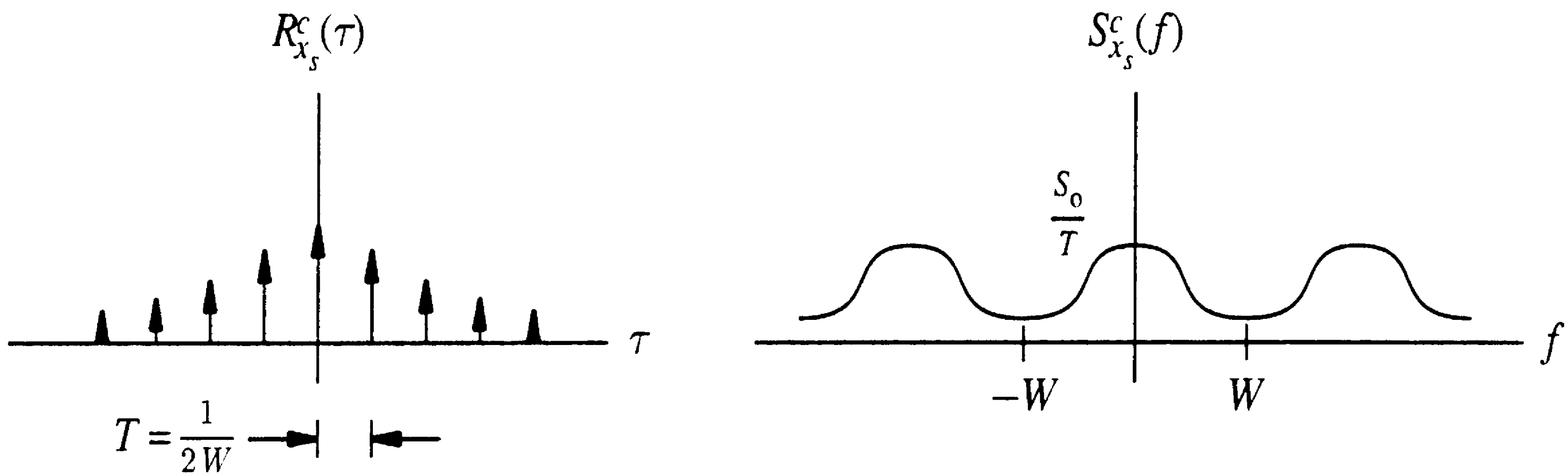
CONTINUOUS CORRELATION FUNCTION





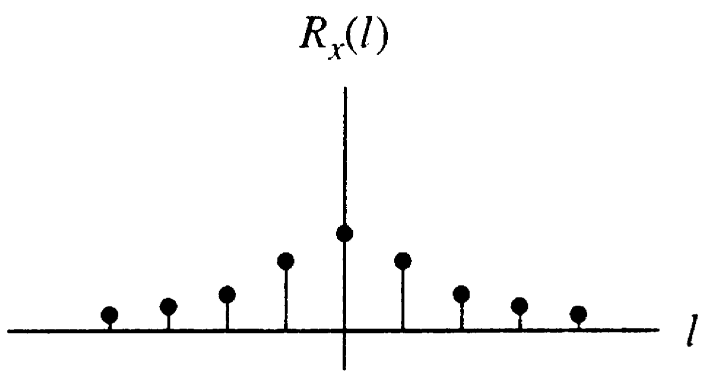
CONTINUOUS AND DISCRETE SPECTRA

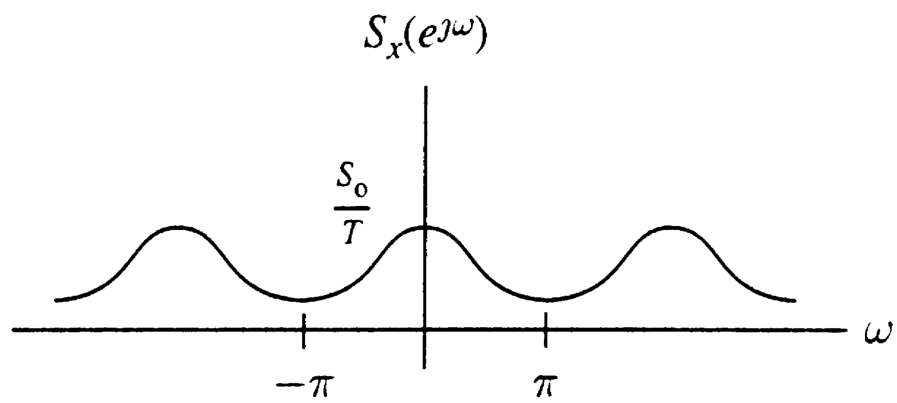
SAMPLED CORRELATION FUNCTION



CONTINUOUS AND DISCRETE SPECTRA

DISCRETE CORRELATION FUNCTION





WIENER PROCESS REVISITED

DENSITY FUNCTION

$$f_{x_c(t)}(\mathsf{x}) = \frac{1}{\sqrt{2\pi\nu_o(t-t_o)}}\; e^{-\frac{\mathsf{x}^2}{2\nu_o(t-t_o)}}$$

Gaussian with mean 0, variance $\nu_o(t-t_o)$

CORRELATION FUNCTION

$$\begin{aligned}
\mathsf{R}^c_{x_c}(t_1,t_2) &= \mathsf{C}^c_{x_c}(t_1,t_2) = \mathcal{E}\{x_c(t_1)x_c(t_2)\} \quad \text{(assume } t_1 > t_2\text{)} \\
&= \underbrace{\mathcal{E}\{x_c(t_2)x_c(t_2)\}}_{\downarrow} + \underbrace{\mathcal{E}\{[x_c(t_1)-x_c(t_2)]\,x_c(t_2)\}}_{\swarrow} \\
&= \nu_o(t_2-t_o) \qquad\qquad\qquad\qquad 0 \quad \text{(independent increments)}
\end{aligned}$$

By a symmetrical argument $\mathsf{R}^c_{x_c}(t_1,t_2) = \nu_o(t_1-t_o)$ for $t_2 \geq t_1$

Therefore . . . $\mathsf{R}^c_{x_c}(t_1,t_2) = \nu_o \min\,(t_1-t_o, t_2-t_o)$

CONTINUOUS GAUSSIAN WHITE NOISE

- White noise is defined as the *derivative* of the Wiener process.

  Approximate the derivative as

$$w'_c(t) \stackrel{\text{def}}{=} \frac{x_c(t+\Delta t) - x_c(t)}{\Delta t}$$

  where $x_c(t)$ is a Wiener process, and compute the correlation function.

GAUSSIAN WHITE NOISE (cont'd.)

$$\mathsf{R}^c_{w'_c}(t_1, t_2) = \mathcal{E}\{w'_c(t_1) w'_c(t_2)\}$$

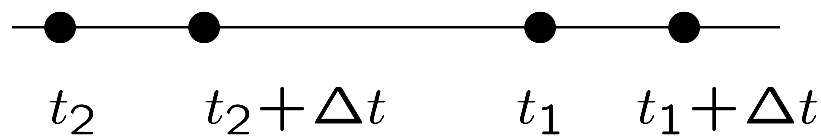
$$= \mathcal{E}\left\{\left[\frac{x_c(t_1+\Delta t) - x_c(t_1)}{\Delta t}\right]\left[\frac{x_c(t_2+\Delta t) - x_c(t_2)}{\Delta t}\right]\right\}$$

$$= \mathcal{E}\left\{\frac{\begin{array}{r} x_c(t_1+\Delta t)x_c(t_2+\Delta t) - x_c(t_1)x_c(t_2+\Delta t) \\ -x_c(t_1+\Delta t)x_c(t_2) + x_c(t_1)x_c(t_2) \end{array}}{(\Delta t)^2}\right\}$$

GAUSSIAN WHITE NOISE (cont'd.)

$$\mathsf{R}^c_{w'_c}(t_1,t_2) = \mathcal{E}\left\{ \frac{\begin{array}{r} x_c(t_1+\Delta t)x_c(t_2+\Delta t) - x_c(t_1)x_c(t_2+\Delta t) \\ -x_c(t_1+\Delta t)x_c(t_2) + x_c(t_1)x_c(t_2) \end{array}}{(\Delta t)^2} \right\}$$

Assume $t_1 > t_2$
and $t_1 > t_2 + \Delta t$:

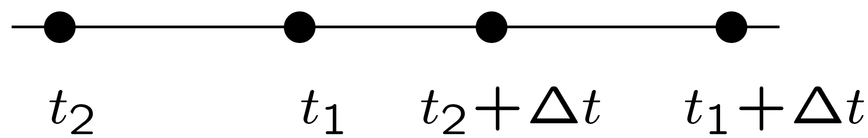


$$\mathsf{R}^c_{w'_c}(t_1,t_2) = \frac{\begin{array}{r} \nu_\mathsf{o}(t_2+\Delta t - t_\mathsf{o}) - \nu_\mathsf{o}(t_2+\Delta t - t_\mathsf{o}) \\ -\nu_\mathsf{o}(t_2 - t_\mathsf{o}) + \nu_\mathsf{o}(t_2 - t_\mathsf{o}) \end{array}}{(\Delta t)^2} = 0$$

GAUSSIAN WHITE NOISE (cont'd.)

$$\mathsf{R}^c_{w'_c}(t_1,t_2) = \mathcal{E}\left\{ \frac{\begin{array}{r} x_c(t_1+\Delta t)x_c(t_2+\Delta t) - x_c(t_1)x_c(t_2+\Delta t) \\ -x_c(t_1+\Delta t)x_c(t_2) + x_c(t_1)x_c(t_2) \end{array}}{(\Delta t)^2} \right\}$$

For $t_1 > t_2$ and
$t_2 \leq t_1 \leq t_2 + \Delta t$:



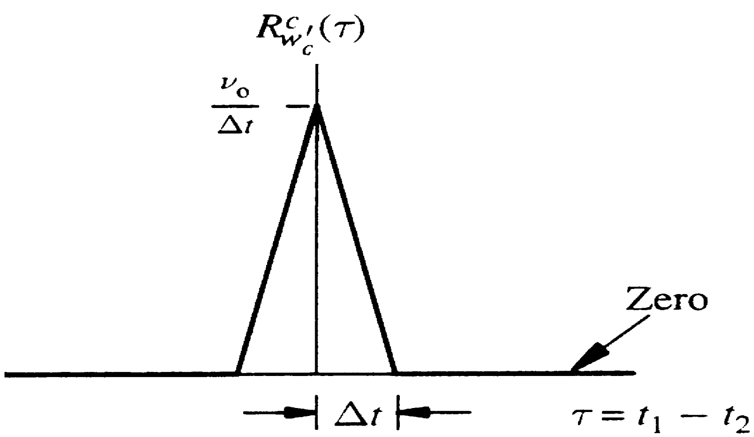
$$\mathsf{R}^c_{w'_c}(t_1,t_2) = \frac{\begin{array}{r} \nu_o(t_2+\Delta t - t_o) - \nu_o(t_1 - t_o) \\ -\nu_o(t_2 - t_o) + \nu_o(t_2 - t_o) \end{array}}{(\Delta t)^2} = \frac{\nu_o}{\Delta t}\left(1 - \frac{t_1 - t_2}{\Delta t}\right)$$

GAUSSIAN WHITE NOISE (cont'd.)

Let $\tau = t_1 - t_2 > 0$, then

$$\mathrm{R}^c_{w'_c}(t_1, t_2) = R^c_{w'_c}(\tau) = \begin{cases} \dfrac{\nu_0}{\Delta t}\left(1 - \dfrac{\tau}{\Delta t}\right) & \tau < \Delta t \\ 0 & \tau > \Delta t \end{cases}$$

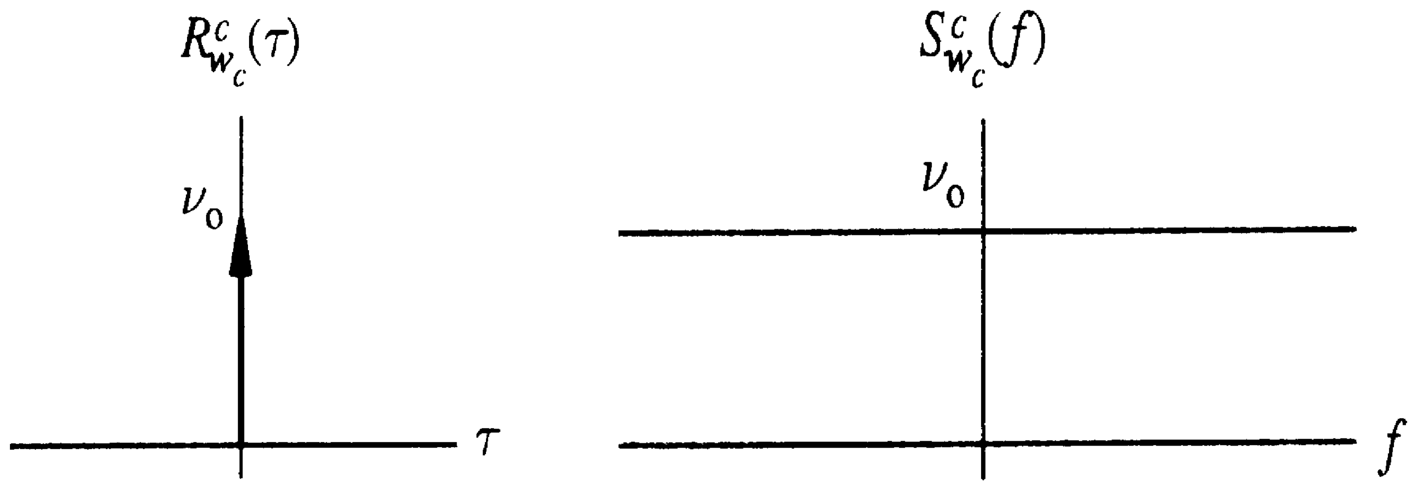
A similar argument applies for $\tau < 0$ $(t_2 > t_1)$.



$$\text{Area} = \frac{1}{2}\frac{\nu_0}{\Delta t} \cdot 2\Delta t = \nu_0$$

DEFINITION OF WHITE NOISE

White noise $w_c(t)$ is the limiting case of $w'_c(t)$ as $\Delta t \to 0$



- White noise has *infinite* power, $R_{w_c}(0) = \infty$.

COMMENTS ON WHITE NOISE

- Continuous white noise is a fictitious process.
- Derivative of the Wiener process does not really exist.
- White noise is observed only after it passes through a system (filter).
- White noise is useful and convenient for engineering analysis.
- It is defined to be Gaussian for mathematical consistency.

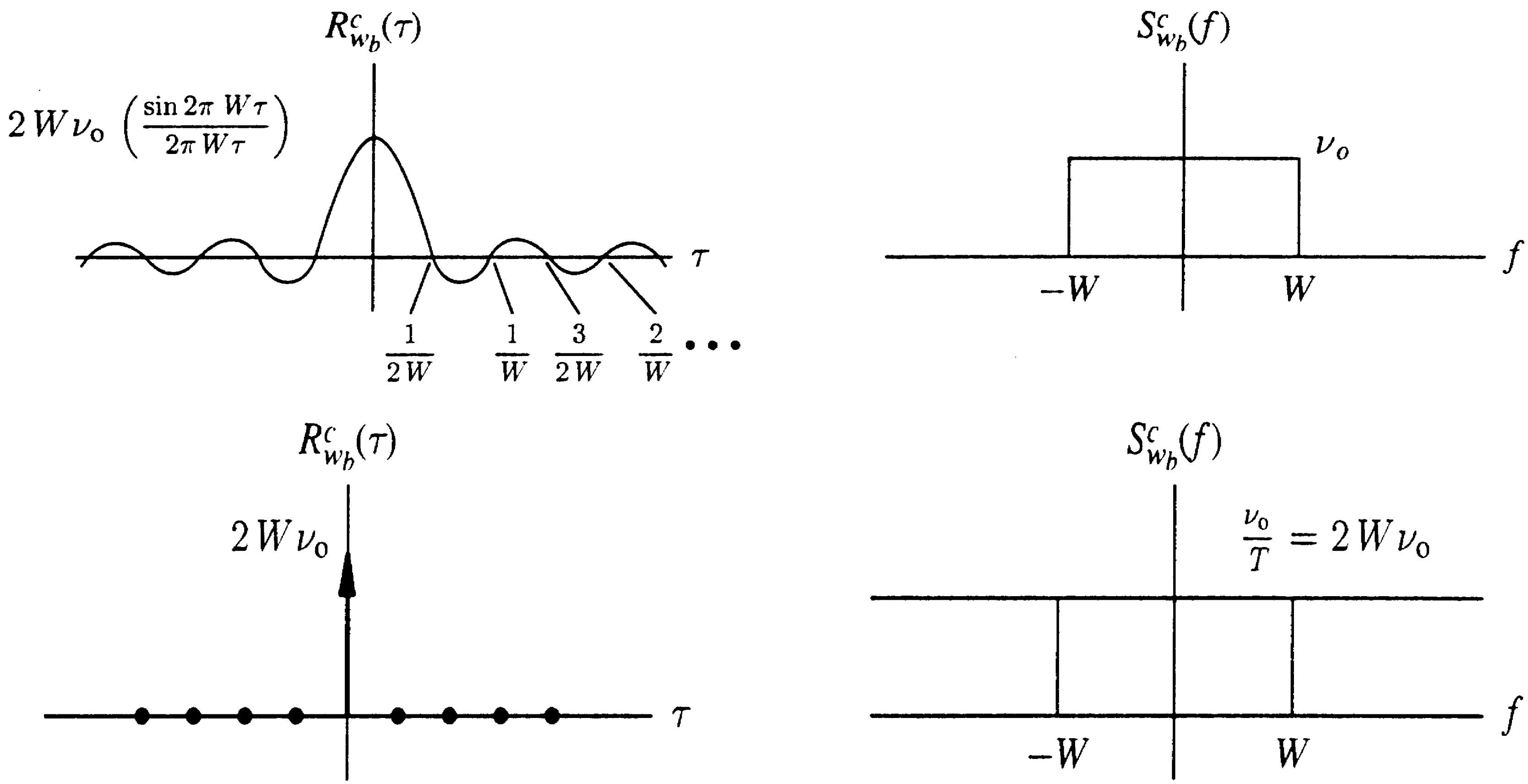
SAMPLING WHITE NOISE

- White noise is *not* bandlimited; therefore it cannot be reconstructed from samples.

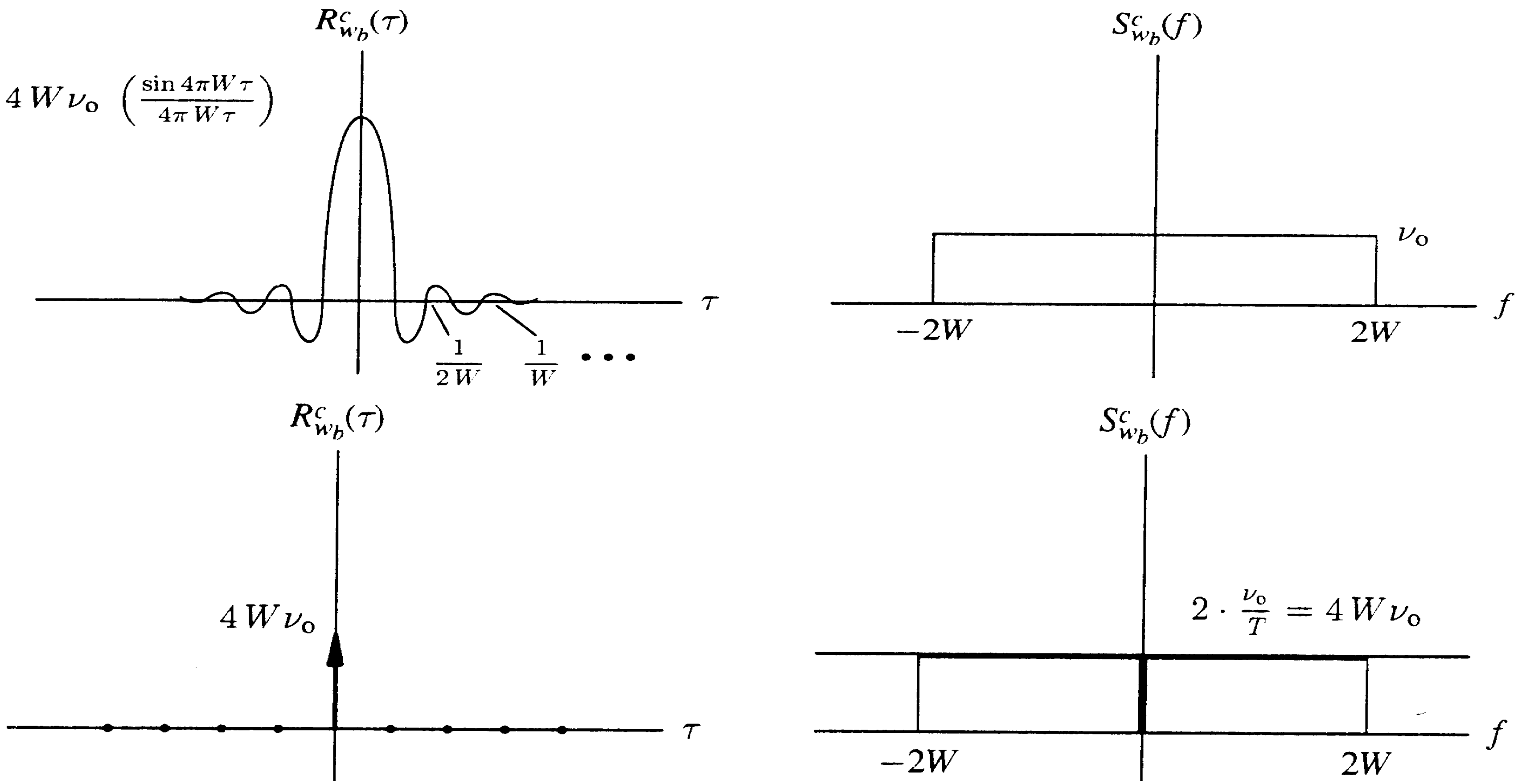
- For most purposes involving sampled signals, we can replace ∞ bandwidth white noise by *bandlimited* white noise.

- Bandlimited white noise has to be defined carefully.

SAMPLING BANDLIMITED WHITE NOISE



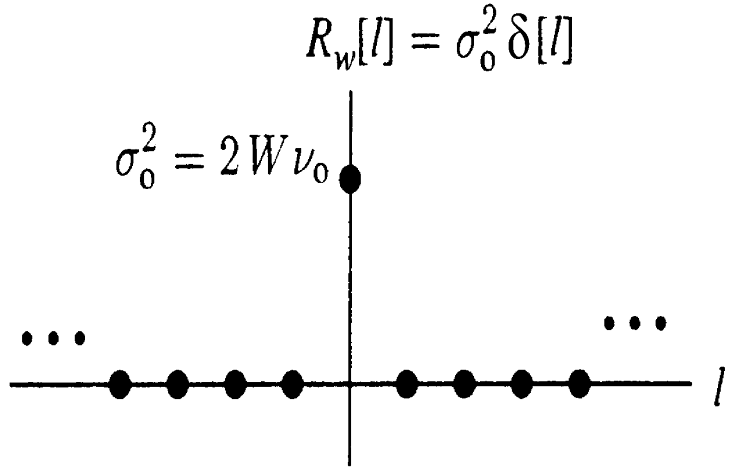
UNDERSAMPLING BANDLIMITED NOISE

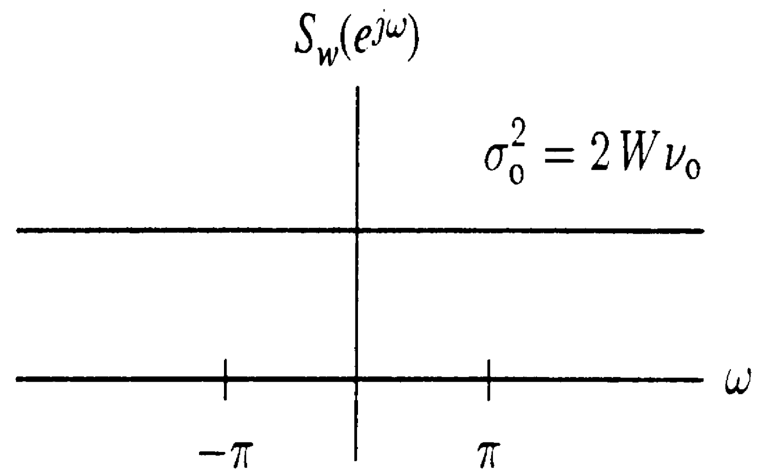


SAMPLED BANDLIMITED WHITE NOISE

DISCRETE REPRESENTATION

(Correctly sampled)

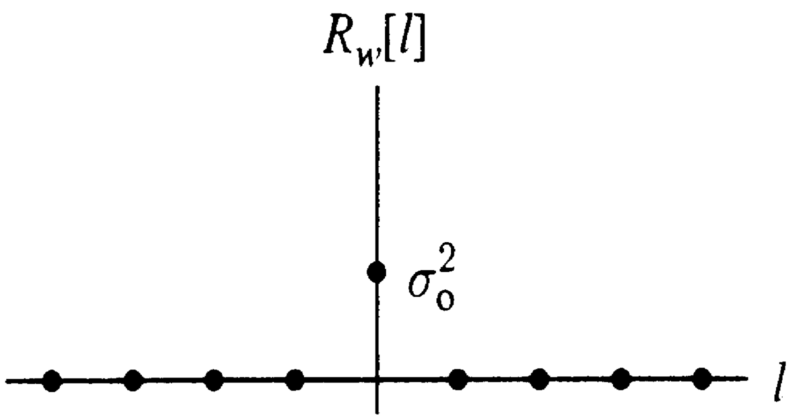


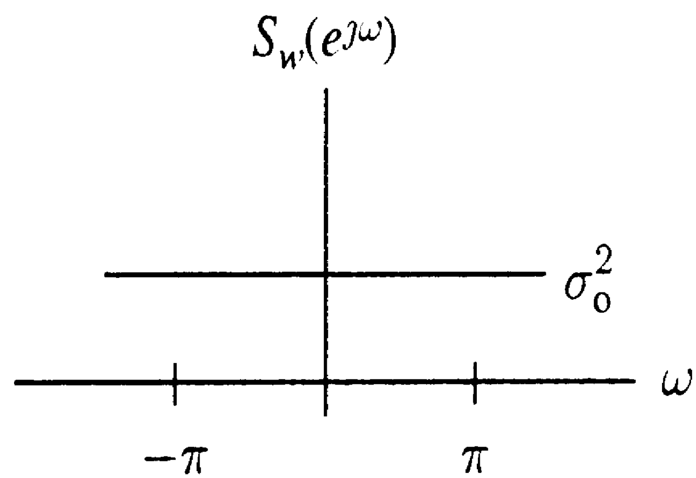


DISCRETE WHITE NOISE

DEFINITION

Any zero-mean process with $R_w[l] = \sigma_o^2 \delta[l]$ is called a (discrete) white noise process.





DISCRETE WHITE NOISE (cont'd.)

CORRELATION MATRIX

$$\mathbf{R}_w = \begin{bmatrix} \sigma_o^2 & 0 & \cdots & 0 \\ 0 & \sigma_o^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_o^2 \end{bmatrix} = \sigma_o^2 \mathbf{I}$$

- Discrete white noise does not have to be Gaussian.

DISCRETE WHITE NOISE (cont'd.)

- For Gaussian discrete white noise $\mathbf{w}^{*T}\mathbf{C}_w^{-1}\mathbf{w} = \frac{\|\mathbf{w}\|^2}{\sigma_o^2}$; therefore

$$f_w(\mathbf{w}) \;=\; \frac{1}{(2\pi\sigma_o^2)^{N/2}}\, e^{-\frac{\|\mathbf{w}\|^2}{2\sigma_o^2}} \qquad \text{(real case)}$$

$$f_w(\mathbf{w}) \;=\; \frac{1}{(\pi\sigma_o^2)^{N}}\, e^{-\frac{\|\mathbf{w}\|^2}{\sigma_o^2}} \qquad \text{(complex case)}$$

- Complex Gaussian white noise has real, imaginary parts jointly Gaussian. The magnitude is Rayleigh distributed; the phase is uniform.