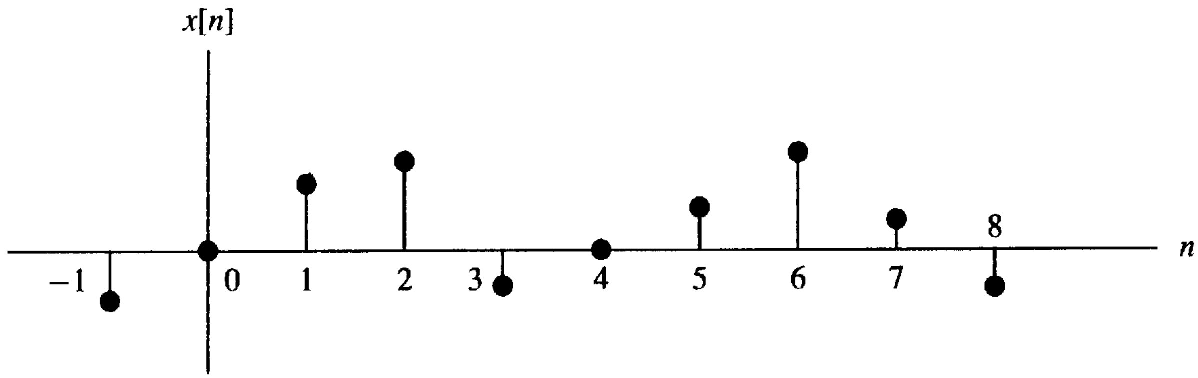
RANDOM SIGNALS/SEQUENCES



$\cdots$, $x[-1]$, $x[0]$, $x[1]$, $\cdots$ are random variables

- Sequence is called a *random signal* or *time series*.

- Underlying model is a *random process* or *stochastic process*.

NOTES ABOUT RANDOM SIGNALS

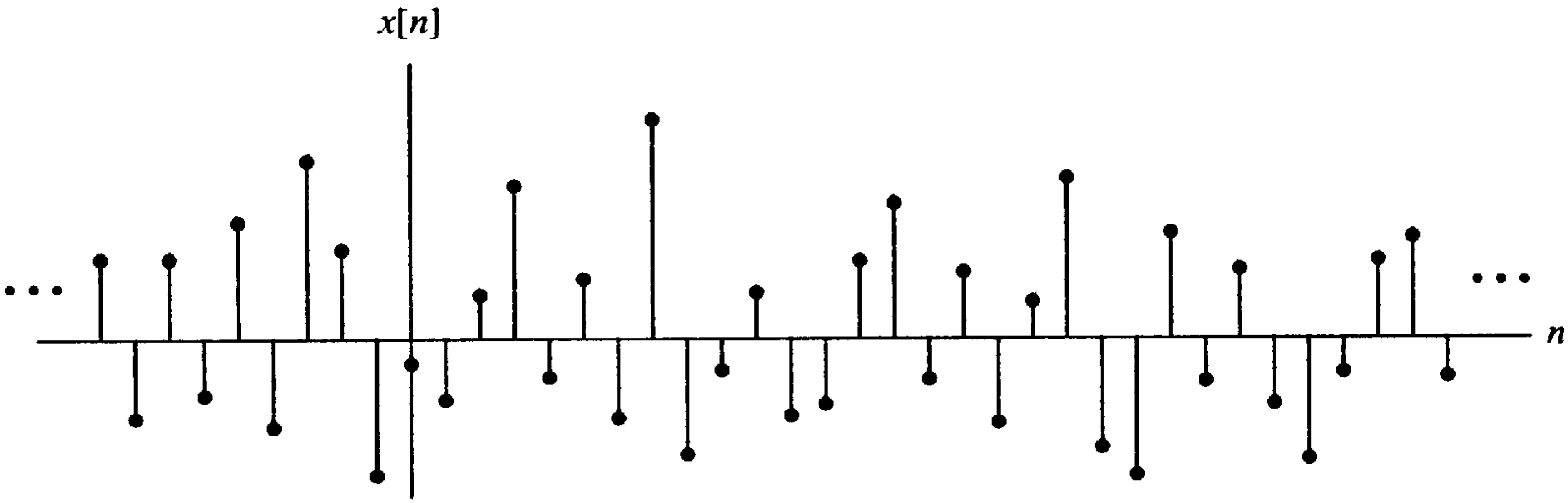
- Random signals generally have *infinite* energy:

$$\mathcal{E}\left\{\sum_{n=-\infty}^{\infty}|x[n]|^2\right\} = \sum_{n=-\infty}^{\infty}\mathcal{E}\left\{|x[n]|^2\right\} = \infty$$

- Random signals may be *predictable*.

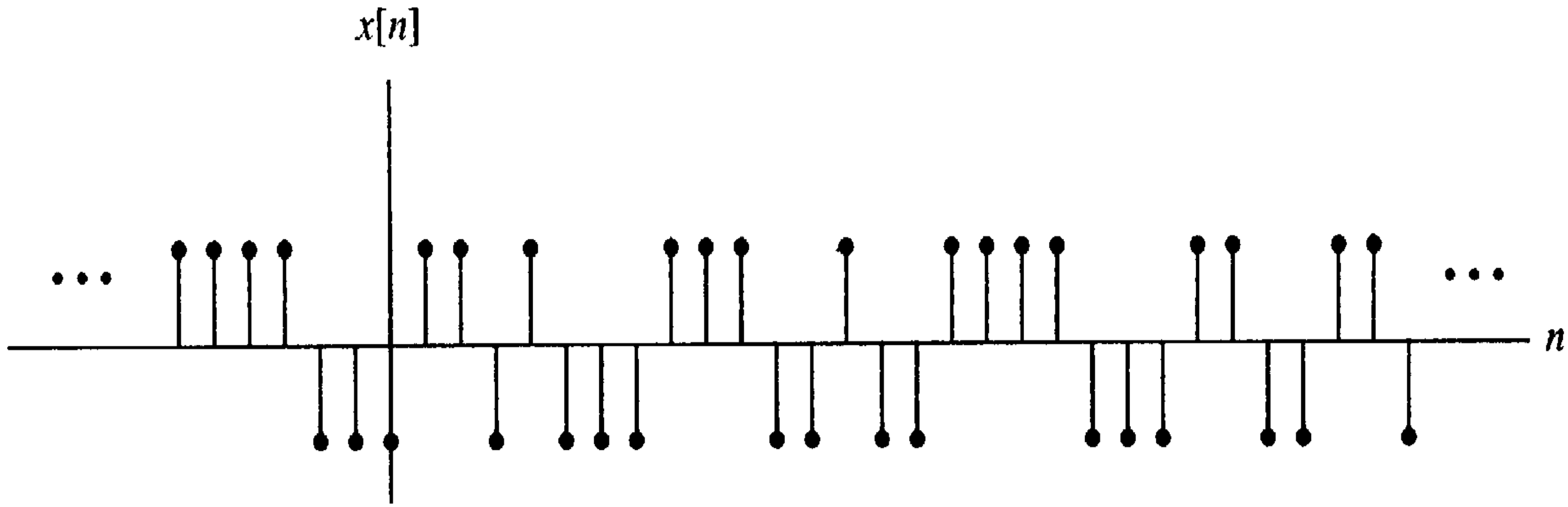
RANDOM SIGNAL EXAMPLES

NOISE



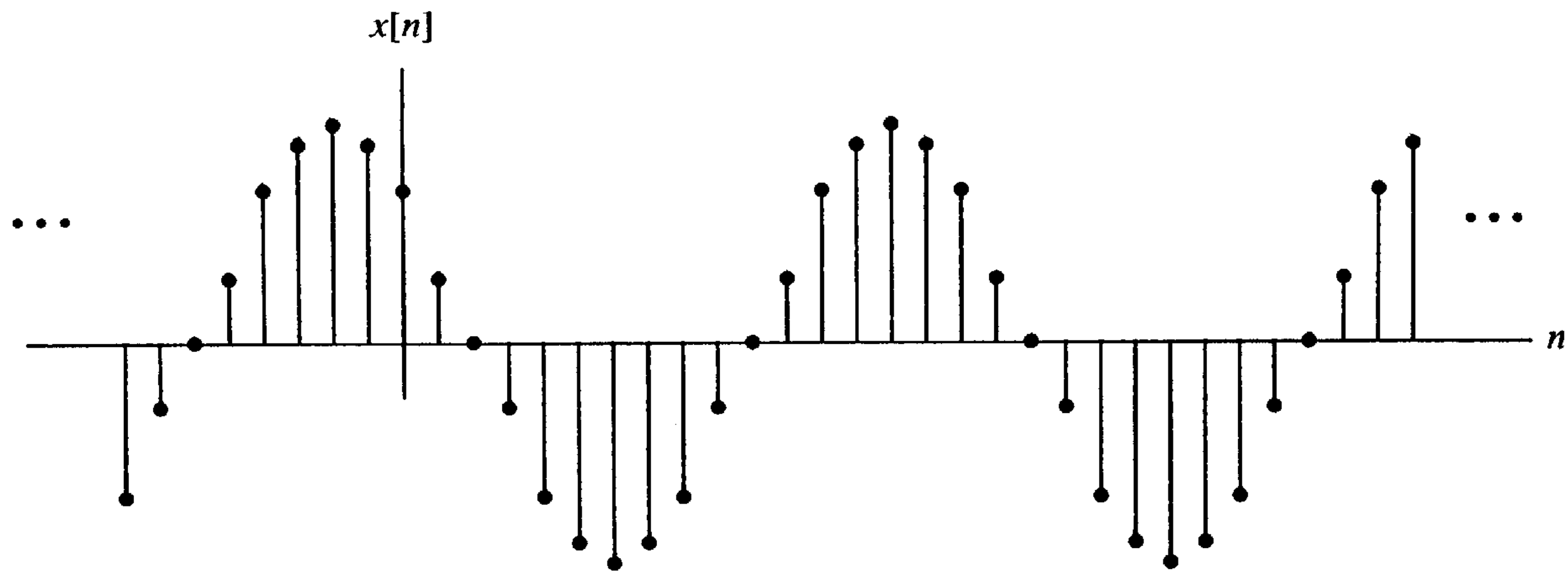
RANDOM SIGNAL EXAMPLES (cont'd.)

BINARY CODED DATA



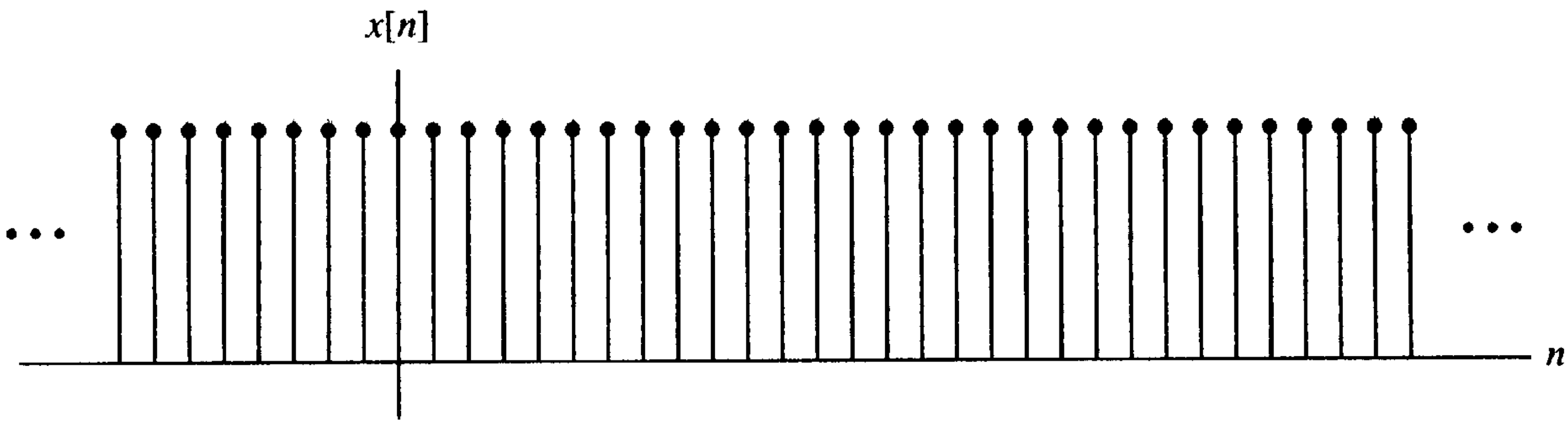
RANDOM SIGNAL EXAMPLES (cont'd.)

RANDOM SINUSOID



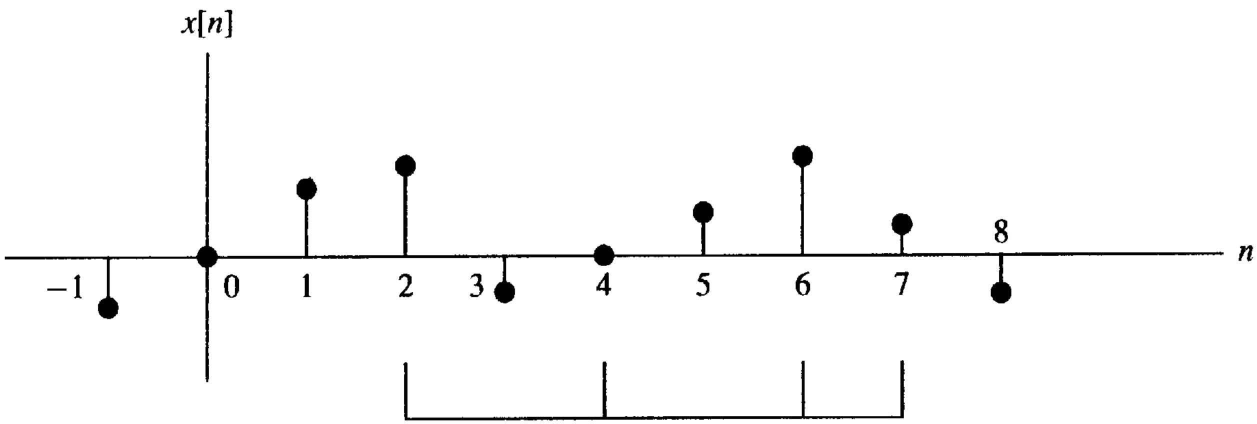
RANDOM SIGNAL EXAMPLES (cont'd.)

BATTERY VOLTAGE



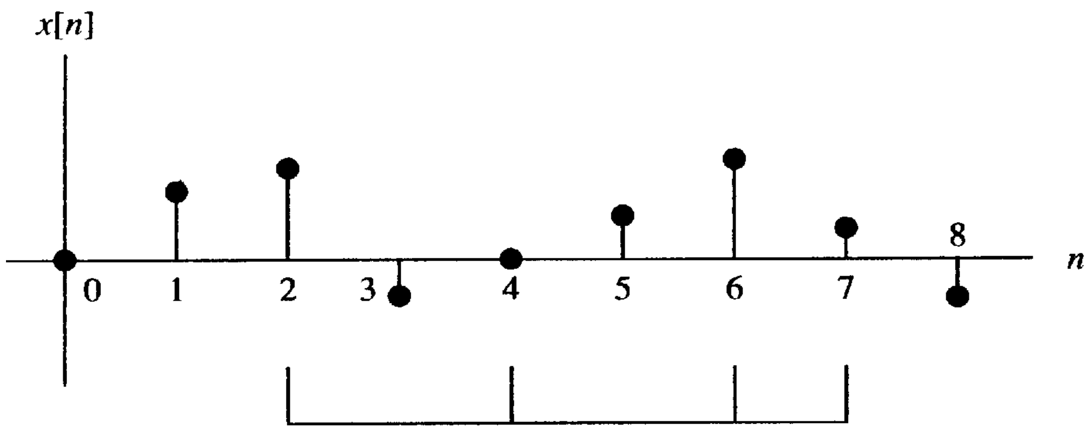
STATISTICAL CHARACTERIZATION OF RANDOM SIGNALS

- Random signals are characterized by the joint distribution or joint density for the samples.



STATISTICAL CHARACTERIZATION (cont'd.)

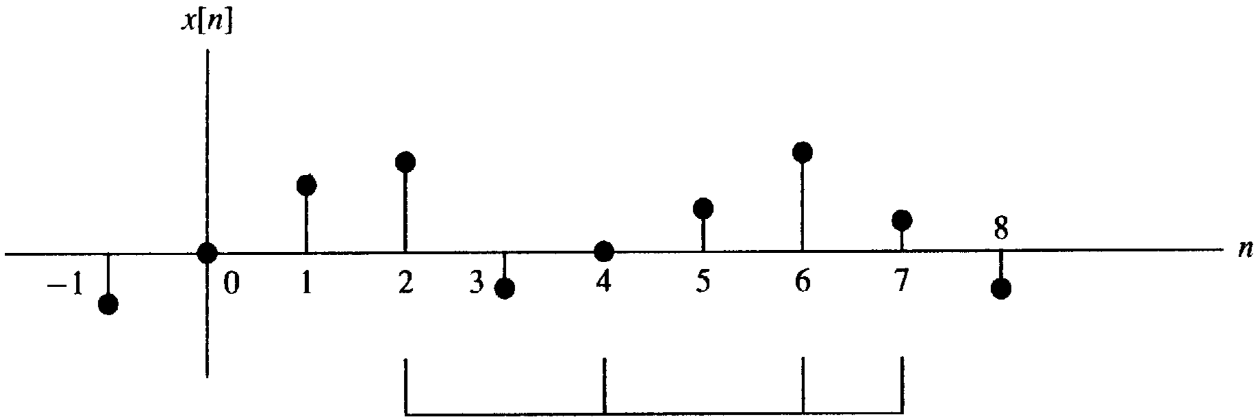
- It is necessary and sufficient to consider just blocks of contiguous samples, represented as random vectors:



$$\boldsymbol{x} = \begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n+N-1] \end{bmatrix}$$

STATIONARITY (STRICT SENSE)

A random process is stationary if any joint density or distribution function depends only on the *spacing between samples* and not on where in the sequence the samples occur.



$$f_{x[2]x[4]x[6]x[7]} = f_{x[1]x[3]x[5]x[6]} = f_{x[-1]x[1]x[3]x[4]} = \cdots$$

STATIONARITY (cont'd.)

- Stationarity requires that all moments

  $$\mathcal{E}\left\{x^{k_0}[n_0] \cdot x^{k_1}[n_1] \ldots x^{k_L}[n_L]\right\}$$

  depend only on the intersample spacing.

  Equivalently:

  $$\mathcal{E}\left\{x^{k_0}[n] \cdot x^{k_1}[n+l_1] \cdots x^{k_L}[n+l_L]\right\}$$

  is <u>not</u> a function of n.

ENSEMBLE CONCEPT

- A random variable arises as the outcome of an experiment.

  Example: Roll of a die

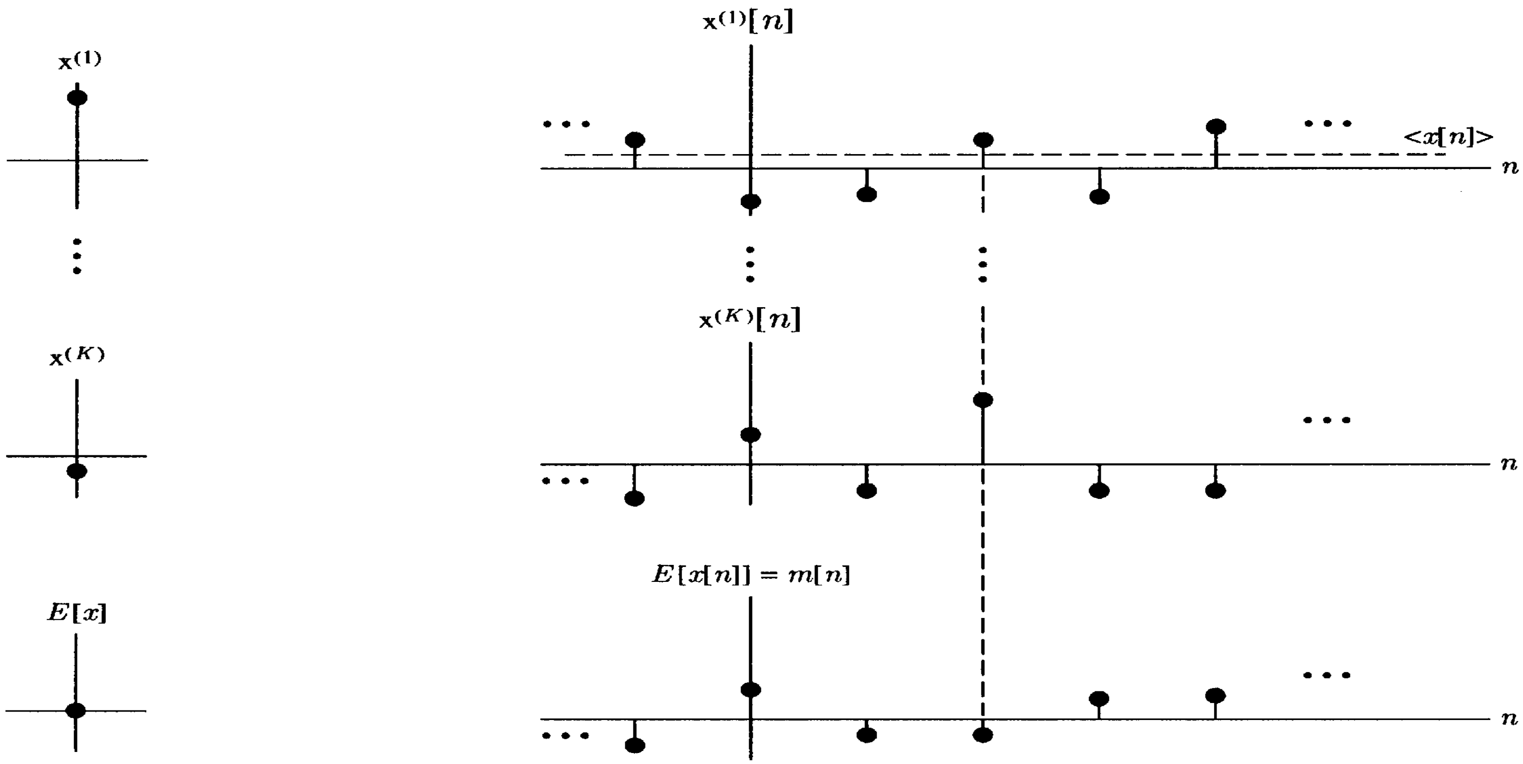
- A random sequence is also the outcome of an experiment.

  Example: Record of binary data

- The collection of all possible outcomes is called an *ensemble*.

ILLUSTRATION OF ENSEMBLE

RANDOM VARIABLE RANDOM PROCESS



ENSEMBLE AVERAGES

RANDOM VARIABLE

$$\mathsf{m}_x = \mathcal{E}\{x\} = \int_{-\infty}^{\infty} \mathsf{x}\, f_x(\mathsf{x}) d\mathsf{x} \;\approx\; \frac{1}{K}\sum_{k=1}^{K} x^{(k)}$$

RANDOM PROCESS

$$\mathsf{m}_x[n] = \mathcal{E}\{x[n]\} = \int_{-\infty}^{\infty} \mathsf{x}_n\, f_{x[n]}(\mathsf{x}_n) d\mathsf{x}_n \;\approx\; \frac{1}{K}\sum_{k=1}^{K} x^{(k)}[n]$$

- As $K \longrightarrow \infty$ statistical expectation is equivalent to averaging down the ensemble.

SIGNAL AVERAGES

- For a single experimental outcome (ensemble member):

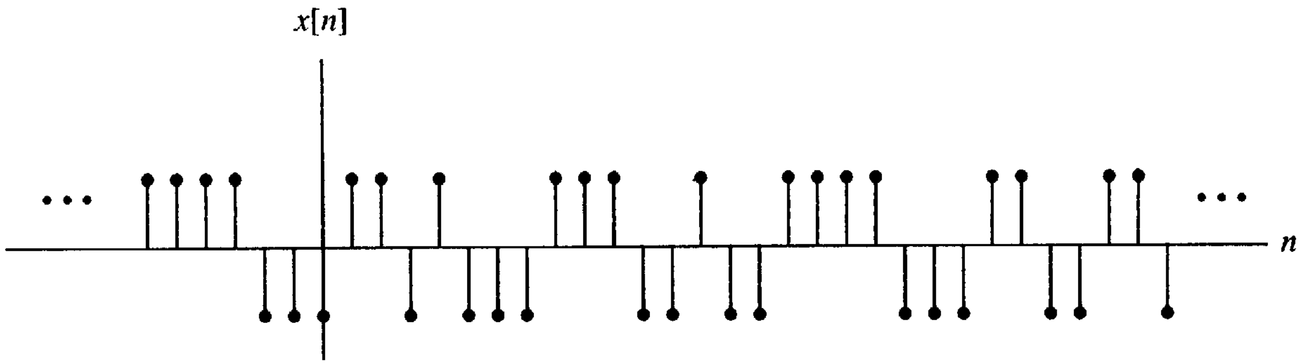
$$\langle x[n] \rangle \stackrel{\text{def}}{=} \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]$$

- A stationary process is <u>*ergodic*</u> if signal averages are equal to ensemble averages with probability 1:

$$\left\langle x^{k_0}[n]\, x^{k_1}[n+l_1] \cdots x^{k_L}[n+l_L] \right\rangle$$

$$\doteq \mathcal{E}\left\{ x^{k_0}[n]\, x^{k_1}[n+l_1] \cdots x^{k_L}[n+l_L] \right\}$$

BERNOULLI PROCESS



Samples are independent and

$$x[n] = \begin{cases} +1 & \text{with probability P} \\ -1 & \text{with probability } 1-\text{P} \end{cases}$$

- For $\text{P} = \frac{1}{2}$ the process is called *binary white noise*.

BERNOULLI PROCESS

PROBABILISTIC DESCRIPTION

One can write the probability of any subsequence explicitely.

Example:

$$\Pr\Big[x[0] = 1,\ x[1] = 1,\ x[2] = -1,\ x[3] = 1,\ x[4] = -1\Big]$$

$$= \mathsf{P} \cdot \mathsf{P} \cdot (1 - \mathsf{P}) \cdot \mathsf{P} \cdot (1 - \mathsf{P}) = \mathsf{P}^3 (1 - \mathsf{P})^2$$

- Since the probability of any sequence is independent of the starting point, the process is *stationary*.

MOMENTS OF THE BERNOULLI PROCESS

MEAN

$$\mathcal{E}\{x[n]\} = \mathsf{P} \cdot (+1) + (1 - \mathsf{P})(-1) = 2\mathsf{P} - 1$$

VARIANCE

$$\begin{aligned}
\mathrm{Var}\Big[x[n]\Big] &= \mathcal{E}\Big\{(x[n])^2\Big\} - (\mathcal{E}\{x[n]\})^2 \\
&= \mathsf{P} \cdot (+1)^2 + (1 - \mathsf{P}) \cdot (-1)^2 - (2\mathsf{P} - 1)^2 \\
&= 1 - (2\mathsf{P} - 1)^2 = 4\mathsf{P}(1 - \mathsf{P})
\end{aligned}$$

BERNOULLI PROCESS

SUMMARY

	Bernoulli Process	Binary White Noise
Distribution	$\Pr[x[n]=+1] \;=\; \mathrm{P}$ $\Pr[x[n]=-1] \;=\; 1-\mathrm{P}$	$\Pr[x[n]=+1] \;=\; \frac{1}{2}$ $\Pr[x[n]=-1] \;=\; \frac{1}{2}$
Mean	$2\mathrm{P}-1$	0
Variance	$4\mathrm{P}(1-\mathrm{P})$	1

RANDOM WALK

Consider $x[n] = \sum_{k=-\infty}^{n} \xi[k]$ where $\xi[k]$ a Bernoulli process.

- $x[n]$ can take on any integer values.
- By Central Limit Theorem $x[n]$ is Gaussian (∞ variance).
- $x[n]$ has *independent increments*:

$$(x[1] - x[0]) \; ; \; (x[2] - x[1]) \; ; \; \cdots ; \; (x[k] - x[k-1])$$

$\Rightarrow$ <u>Events defined on non-overlapping time intervals are independent</u>.

RANDOM WALK (cont'd.)

The process $x_{n_0}[n] = x[n] - x[n_0]$; $n > n_0$ is called a *random walk*:

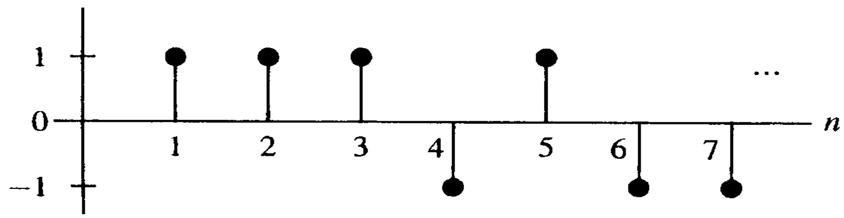
$$x_{n_0}[n] = \sum_{k=n_0+1}^{n} \xi[k] \qquad (n_0 \text{ fixed})$$

Usually $n_0 = 0$ or $n_0 = 1$

- For P $= \frac{1}{2}$ ($\xi[k]$ binary white noise) the random walk is called a *discrete Wiener process*.

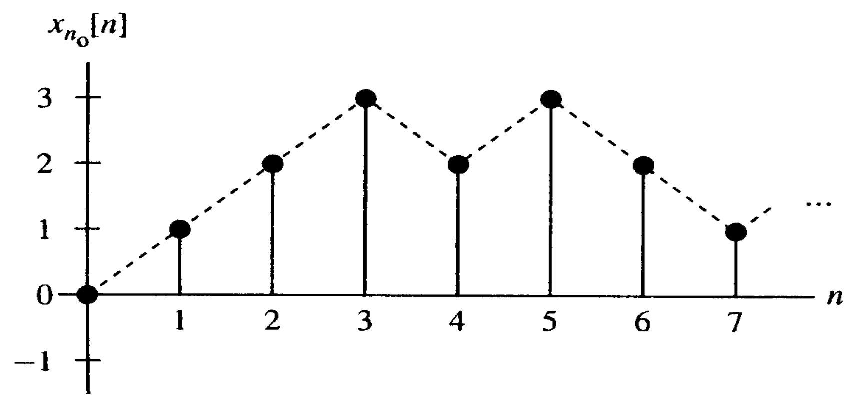
EXAMPLE OF RANDOM WALK

BERNOULLI PROCESS



RANDOM WALK

initial condition: $x_{n_o}[0] = 0$



RANDOM WALK

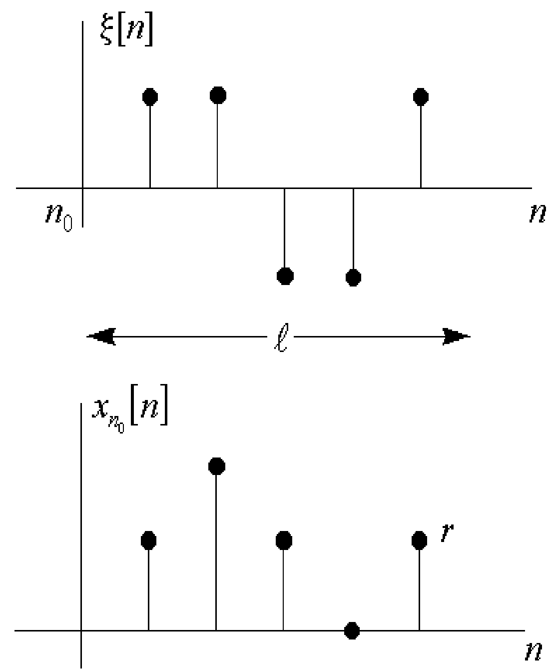
PROBABILISTIC DESCRIPTION

Define $\ell = n - n_o$
(relative time from start)

Suppose there are q $\quad +1$'s
and $l - q$ $\quad -1$'s

then $r = q - (l - q) = 2q - l$

and r can have values:
$-l,\ -l + 2,\ -l + 4,\ \ldots,\ l - 2,\ l$



RANDOM WALK (cont'd.)

Value of the random walk at $n = n_{\text{o}} + l$ is $r = 2q - l$

Number of $+1$'s:

$$q = \frac{l + r}{2}$$

Number of -1's:

$$l - q = \frac{l - r}{2}$$

DISTRIBUTION

$$\Pr\Big[x_{n_{\text{o}}}[n_{\text{o}} + l] = r\Big] = \begin{pmatrix} l \\ q \end{pmatrix} \mathsf{P}^q (1 - \mathsf{P})^{l-q} = \begin{pmatrix} l \\ \frac{l+r}{2} \end{pmatrix} \mathsf{P}^{\frac{l+r}{2}} (1 - \mathsf{P})^{\frac{l-r}{2}}$$

$$r = -l,\ -l + 2,\ \cdots,\ l - 2,\ l$$

MOMENTS OF RANDOM WALK

MEAN

$$\begin{aligned}\mathcal{E}\left\{x_{n_o}[n]\right\} &= \mathcal{E}\left\{\sum_{k=n_o+1}^{n}\xi[k]\right\} = \sum_{k=n_o+1}^{n}\mathcal{E}\left\{\xi[k]\right\} \\ &= (n-n_o)\cdot\mathcal{E}\left\{\xi[k]\right\} = (2\mathrm{P}-1)(n-n_o)\end{aligned}$$

VARIANCE

$$\begin{aligned}\mathrm{Var}\ \left[x_{n_o}[n]\right] &= \sum_{k=n_o+1}^{n}\mathrm{Var}\ \left[\xi[k]\right] \\ &= (n-n_o)\ \mathrm{Var}\ \left[\xi[k]\right] = 4\mathrm{P}(1-\mathrm{P})(n-n_o)\end{aligned}$$

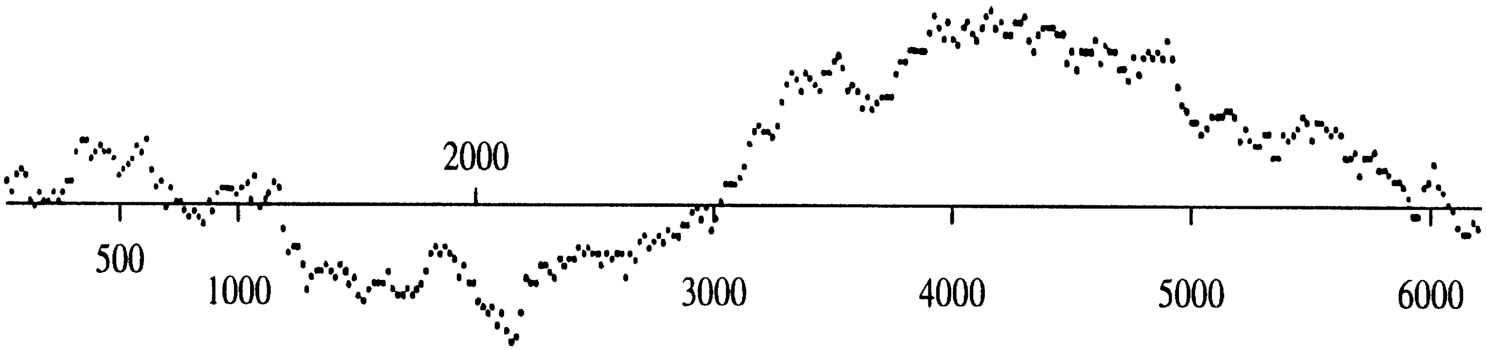
RANDOM WALK

SUMMARY

	Random Walk	Discrete Wiener Process
Distribution $\Pr[x[n] = r]$	$\binom{n-n_\circ}{\frac{n-n_\circ+r}{2}} \mathsf{P}^{\frac{n-n_\circ+r}{2}} (1-\mathsf{P})^{\frac{n-n_\circ-r}{2}}$ $\lvert r\rvert \le n-n_\circ\,;\quad n-n_\circ+r$ even	$\binom{n-n_\circ}{\frac{n-n_\circ+r}{2}} \left(\frac{1}{2}\right)^{n-n_\circ}$ $\lvert r\rvert \le n-n_\circ\,;\quad n-n_\circ+r$ even
Mean	$(2\mathsf{P}-1)(n-n_\circ)$	0
Variance	$4\mathsf{P}(1-\mathsf{P})(n-n_\circ)$	$n-n_\circ$

RANDOM WALK: GENERAL CHARACTER

- Tends to have long runs of positive and negative values.
- Length of runs increases with increasing time, but local behavior remains the same.



PERIODIC RANDOM PROCESS
(STRICT SENSE)

DEFINITION

$$f_{x[n_0],x[n_1],\ldots x[n_L]} = f_{x[n_0+k_0P],\, x[n_1+k_1P],\, \ldots,\, x[n_L+k_LP]}$$

For some ($\exists$) P and for all ($\forall$) choices of the n_i, k_i, and L.

- If the condition is true only for $k_0 = k_1 = \ldots = k_L = k$ then the process is called *cyclostationary*.

PERIODIC RANDOM PROCESS: EXAMPLES

1. $x[n] = A\cos(\omega_0 n + \phi)$ A, ϕ are real random variables

2. $x[n] = \sum_i A_i \cos(\omega_i + \phi_i)$ A_i, ϕ_i are real random variables

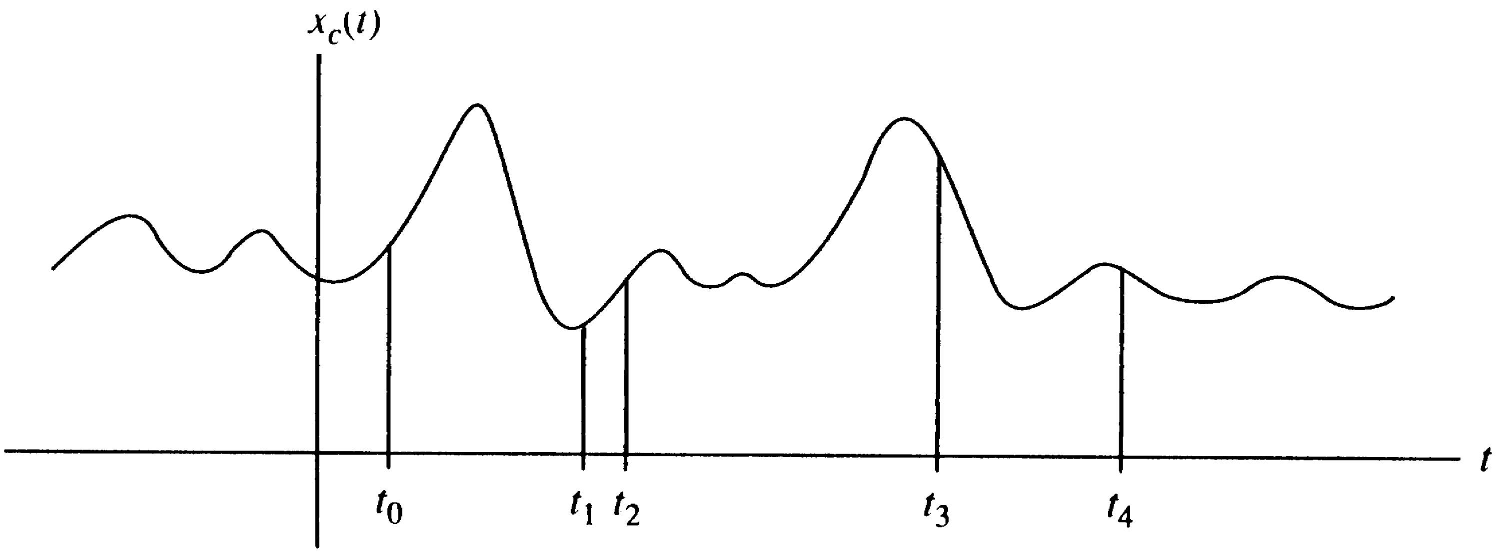
3. $x[n] = \sum_i A_i e^{j\omega_i n}$ $A_i = |A_i|\, e^{j\phi_i}$ and $|A_i|$, ϕ_i are real random variables (A_i is a complex random variable)

- ω_i is assumed to be of the form

$$\omega_i = \frac{K_i}{P_i} 2\pi \quad \text{with} \quad K_i \text{ and } P_i \text{ integers}$$

Otherwise the process is called <u>*almost periodic*</u>.

CONTINUOUS RANDOM PROCESS



- Every sample $x_c(t_i)$ is a random variable.

- Complete statistical description requires being able to specify *every* joint density.

PROPERTIES OF CONTINUOUS RANDOM PROCESSES

STATIONARITY

Distribution/density function depend on only *spacing* between samples.

ERGODICITY

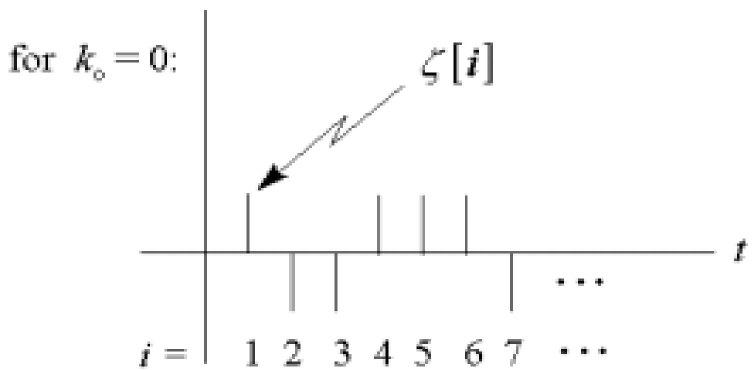
Signal (time) averages $\doteq$ ensemble averages.

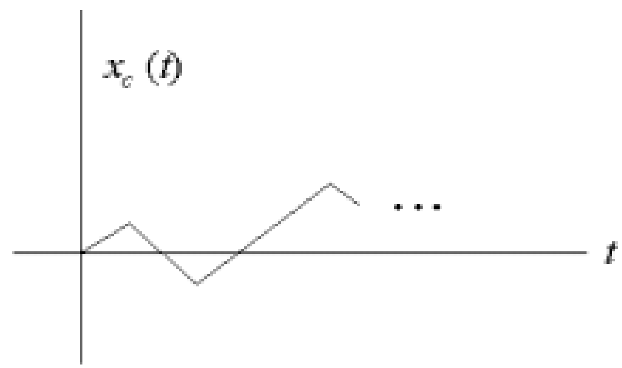
Signal average:

$$< x_c(t) > = \lim_{\Upsilon \to \infty} \frac{1}{2\Upsilon} \int_{-\Upsilon}^{\Upsilon} x_c(t) dt$$

CONTINUOUS WIENER PROCESS (BROWNIAN MOTION PROCESS)

Define $x_c(k\Delta t) = \sum_{i=k_o+1}^{k} s\zeta[i]$ with $s \to 0$, $\Delta t \to 0$, $t = k\Delta t$

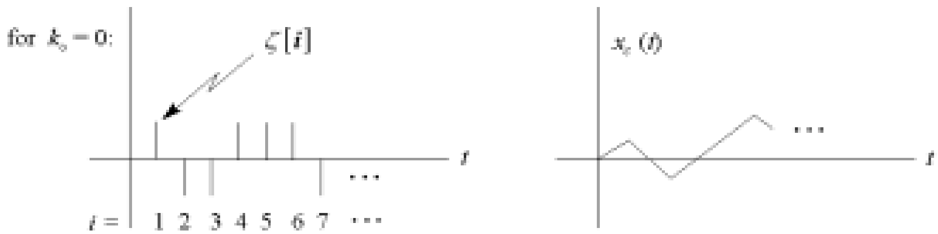




$\zeta[i]$: a binary white noise process

$\mathcal{E}\{x_c(k\Delta t)\} = 0$ Var $[x_c(k\Delta t)] = s^2(k - k_o) = \dfrac{s^2(t - t_o)}{\Delta t}$

WIENER PROCESS (cont'd.)



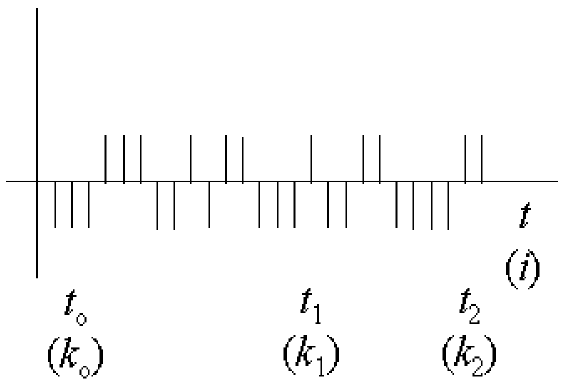
$$\mathcal{E}\{x_c(t)\} = 0 \qquad \text{Var}[x_c(t)] = \frac{s^2(t - t_o)}{\Delta t}$$

Let $\Delta t \to 0$ and $s \to 0$ such that $\dfrac{s^2}{\Delta t} \to \nu_o$ (a constant).

Then $\text{Var}[x_c(t)] \to \nu_o(t - t_o)$ and by the central limit theorem:

$$f_{x_c(t)}(\mathsf{x}) = \frac{1}{\sqrt{2\pi\nu_o(t - t_o)}}\, e^{-\frac{\mathsf{x}^2}{2\nu_o(t-t_o)}} \qquad (x_c \text{ is Gaussian})$$

JOINT DENSITY (TWO SAMPLES)



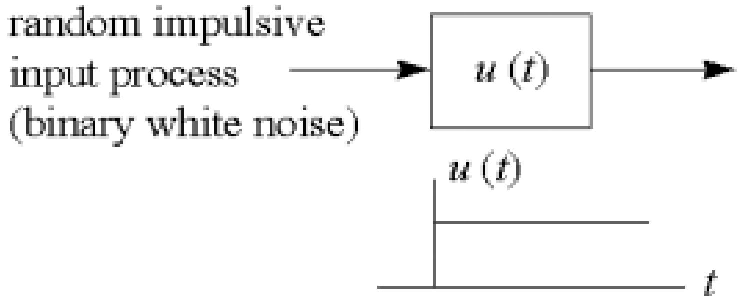
$$\begin{array}{ccccc} x_c(t_2) & = & \underbrace{\sum_{i=k_\circ+1}^{k_1} s\zeta[i]}_{x_c(t_1)} & + & \underbrace{\sum_{i=k_1+1}^{k_2} s\zeta[i]} \\ \downarrow & & \downarrow & & \downarrow \\ x_2 & = & x_1 & + & u \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix} \Rightarrow \; x_1, \; x_2 \text{ jointly Gaussian}$$

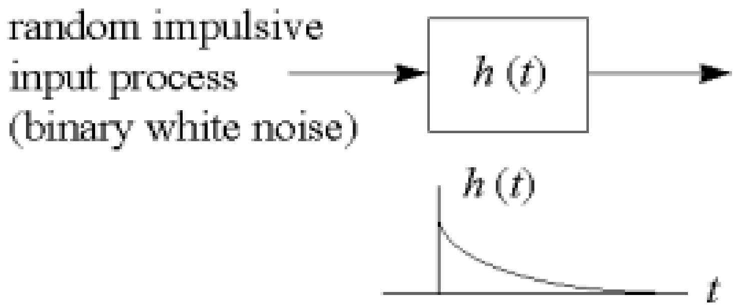
- Argument extends to any number of samples.

  We say the <u>*process is Gaussian*</u>.

CONTINUOUS GAUSSIAN PROCESS



Output is a Wiener process (non-stationary, variance increases linearly with time).



Output is a Gaussian process (stationary, variance is constant) (see Problem 3.25)

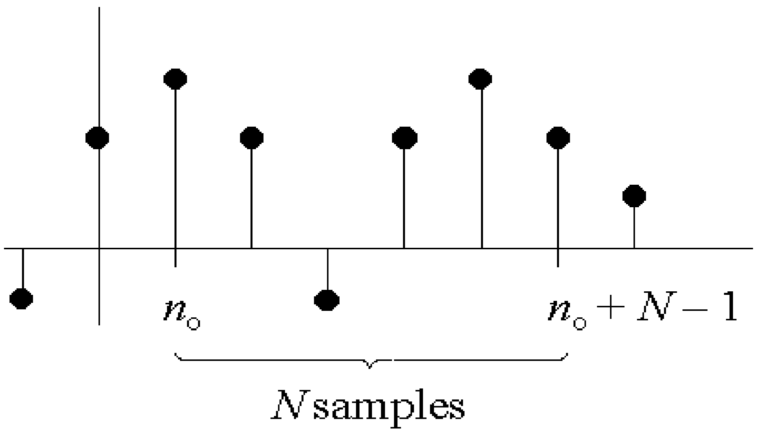
COMPLEX GAUSSIAN PROCESS

- Complex Gaussian processes occur when a random impulsive input is applied to a bandpass filter, and the resulting output process is represented at baseband (see text, Appendix B).

- A complex Gaussian random process has real and imaginary parts which are
  - jointly Gaussian.
  - identically distributed.

DISCRETE GAUSSIAN PROCESS

A discrete random process is Gaussian if any set of N samples are jointly Gaussian.



$$\boldsymbol{x} = \begin{bmatrix} x[n_\text{o}] \\ x[n_\text{o} + 1] \\ \vdots \\ x[n_\text{o} + N - 1] \end{bmatrix}$$

$$f_{\boldsymbol{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} |\mathbf{C}_{\boldsymbol{x}}|^{\frac{1}{2}}} \; e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})^T \mathbf{C}_{\boldsymbol{x}}^{-1} (\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})} \quad \text{(real case)}$$

$$f_{\boldsymbol{x}}(\mathbf{x}) = \frac{1}{\pi^N |\mathbf{C}_{\boldsymbol{x}}|} \; e^{-(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})^{*T} \mathbf{C}_{\boldsymbol{x}}^{-1} (\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})} \quad \text{(complex case)}$$

TRANSFORMATION BY LINEAR SYSTEMS

- A linear transformation of a Gaussian process produces another Gaussian process.