CLASSICAL SPECTRUM ESTIMATION

CORRELATION FUNCTION ESTIMATES

Unbiased Estimate :

$$\widehat{R}'_x[l] = \frac{1}{N_s - l} \sum_{n=0}^{N_s-l-1} x[n+l]x^*[n]$$

$$0 \leq l < N_s$$

$$\mathcal{E}\left\{\widehat{R}'_x[l]\right\} = R_x[l]$$

Biased Estimate :

$$\widehat{R}_x[l] = \frac{1}{N_s} \sum_{n=0}^{N_s-l-1} x[n+l]x^*[n]$$

$$0 \leq l < N_s$$

$$\mathcal{E}\left\{\widehat{R}_x[l]\right\} = \frac{N_s - |l|}{N_s} R_x[l]$$

CORRELOGRAM SPECTRAL ESTIMATE

$$\widehat{S}_x(e^{j\omega}) = \sum_{l=-L}^{L} \widehat{R}_x[l]e^{-j\omega l} \qquad \text{choose} \quad L \leq 10\% N_s$$

THE PERIODOGRAM

If $L = N_s - 1$ then $\widehat{S}_x(e^{j\omega})$ becomes:

$$\sum_{l=-N_s+1}^{N_s-1} \widehat{R}_x[l]e^{-\jmath\omega l} = \frac{1}{N_s}|X(e^{\jmath\omega})|^2$$

where $X(e^{\jmath\omega}) = \sum_{n=0}^{N_s-1} x[n]e^{-\jmath\omega n}$ (see Prob. 10.1).

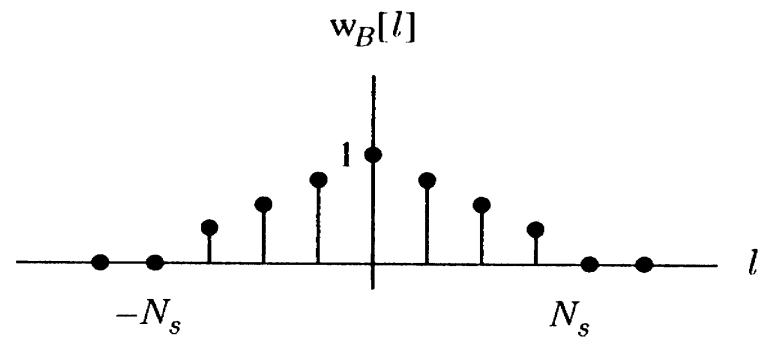
This defines the <u>*periodogram*</u> spectral estimate:

$$\widehat{P}_x(e^{\jmath\omega}) = \frac{1}{N_s}|X(e^{\jmath\omega})|^2$$

MEAN OF THE PERIODOGRAM

$$\mathcal{E}\left\{\widehat{P}_x(e^{\jmath\omega})\right\} = \sum_{l=-N_s+1}^{N_s-1} \mathcal{E}\left\{\widehat{R}_x[l]\right\} e^{-\jmath\omega l} = \sum_{l=-N_s+1}^{N_s-1} \frac{N_s - |l|}{N_s} R_x[l] e^{-\jmath\omega l}$$

$$\mathcal{E}\left\{\widehat{P}_x(e^{\jmath\omega})\right\} = \sum_{l=-\infty}^{\infty} \mathrm{w}_B[l] R_x[l] e^{-\jmath\omega l}$$



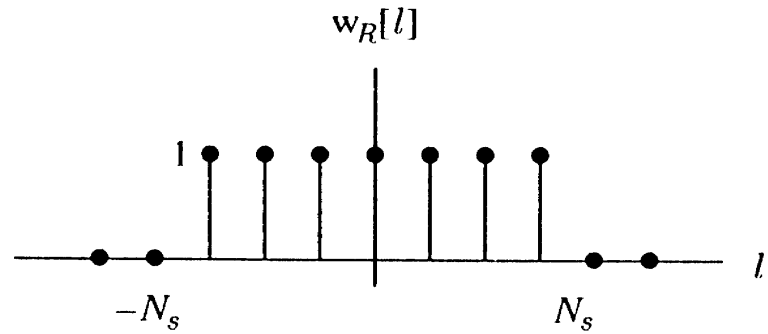
$$\mathcal{E}\left\{\widehat{P}_x(e^{\jmath\omega})\right\} = \frac{1}{2\pi} \mathrm{W}_B(e^{\jmath\omega}) \circledast S_x(e^{\jmath\omega}); \quad \mathrm{W}_B(e^{\jmath\omega}) = \frac{1}{N_s} \frac{\sin^2(N_s\omega/2)}{\sin^2(\omega/2)}$$

- $\widehat{P}_x(e^{\jmath\omega})$ is <u>*biased*</u> but asymptotically unbiased.

SPECTRAL ESTIMATE USING UNBIASED CORRELATION ESTIMATE

$$\widehat{S}'_x(e^{\jmath\omega}) = \sum_{l=-N_s+1}^{N_s-1} \widehat{R}'_{x'}[l]e^{-\jmath\omega l} \Rightarrow \mathcal{E}\left\{\widehat{S}'_x(e^{\jmath\omega})\right\} = \sum_{l=-N_s+1}^{N_s-1} R_x[l]e^{-j\omega l}$$

$$\mathcal{E}\left\{\widehat{S}'_x(e^{\jmath\omega})\right\} = \sum_{l=-\infty}^{\infty} \mathrm{w}_R[l]R_x[l]e^{-j\omega l}$$



$$\mathcal{E}\left\{\widehat{S}'_x(e^{\jmath\omega})\right\} = \frac{1}{2\pi}\mathrm{W}_R(e^{\jmath\omega}) \circledast S_x(e^{\jmath\omega}) \quad \mathrm{W}_R(e^{\jmath\omega}) = \frac{\sin\left((2N_s-1)\omega/2\right)}{\sin\omega/2}$$

- This estimate can be <u>*negative*</u>!

COVARIANCE OF THE PERIODOGRAM

COMPLEX WHITE GAUSSIAN PROCESS (power σ_w^2)

$$\mathrm{Var}\left[\hat{P}_w(e^{\jmath\omega})\right] = \sigma_w^4$$

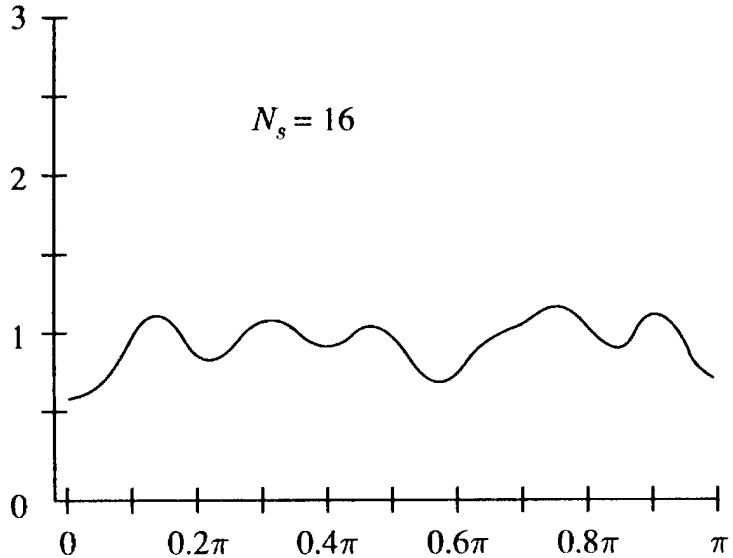
$$\mathrm{Cov}\left[\hat{P}_w(e^{\jmath\omega}), \hat{P}_w(e^{\jmath\omega_2})\right] = \sigma_w^4 \left(\frac{\sin[N_s(\omega_1-\omega_2)/2]}{N_s \sin[(\omega_1-\omega_2)/2]}\right)^2$$

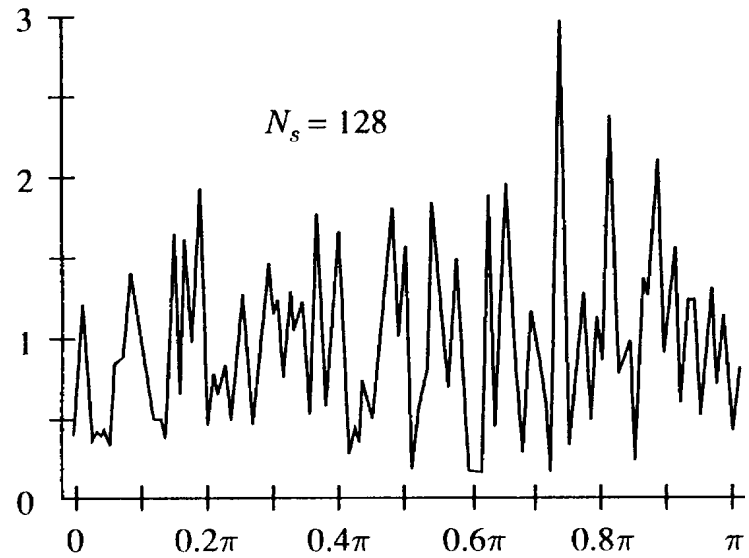
GENERAL GAUSSIAN PROCESS (complex)

$$\mathrm{Var}\ \left[\hat{P}_x(e^{\jmath\omega})\right] \approx S_x^2(e^{\jmath\omega})$$

$$\mathrm{Cov}\ \left[\hat{P}_x(e^{\jmath\omega_1}),\ \hat{P}_x(e^{\jmath\omega_2})\right] \approx S_x(e^{\jmath\omega_1})S_x(e^{\jmath\omega_2})\ \left(\frac{\sin\left[N_s(\omega_1-\omega_2)/2\right]}{N_s \sin\left[(\omega_1-\omega_2)/2\right]}\right)^2$$

BEHAVIOR OF PERIODOGRAM WITH INCREASING LENGTH OF DATA





$$\mathrm{Var}[\hat{P}_\omega(e^{\jmath\omega})] = \sigma_w^4\,; \quad \mathrm{Cov}[\hat{P}_w(e^{\jmath\omega_1}), \hat{P}_w(e^{\jmath\omega_2})] = \sigma_w^4 \left(\frac{\sin[N_s(\omega_1 - \omega_2)/2]}{N_s \sin[(\omega_1 - \omega_2)]} \right)^2$$

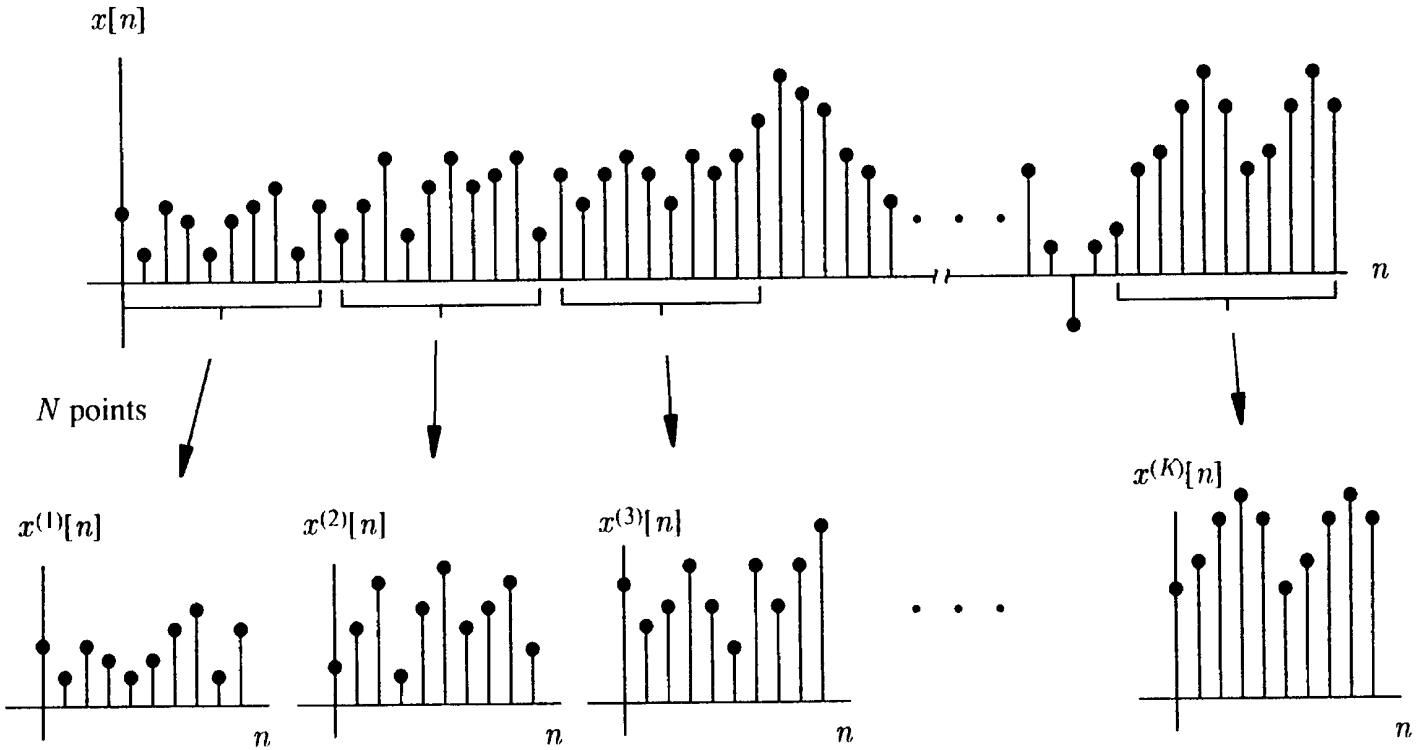
PROCEDURES TO REDUCE VARIANCE

- Bartlett: average periodograms

- Blackman-Tukey: windowing/smoothing

- Welch: combined windowing and averaging

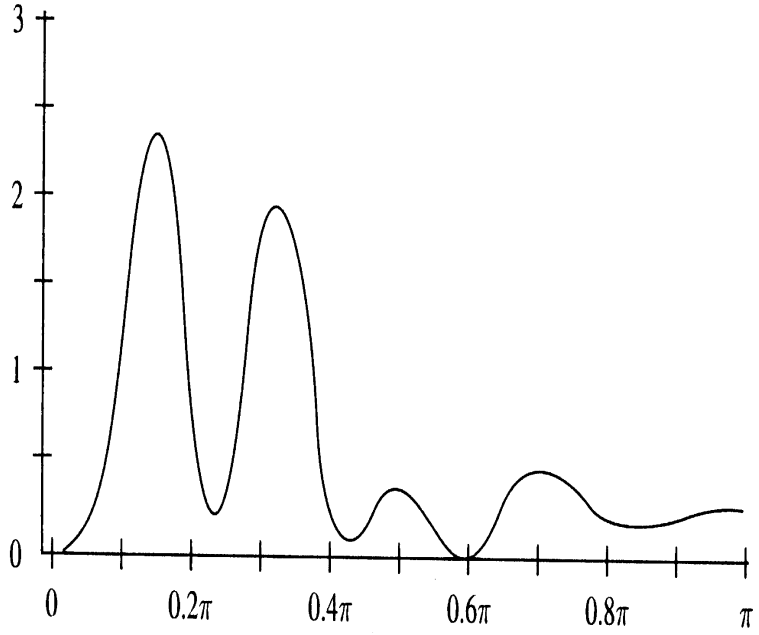
BARTLETT METHOD



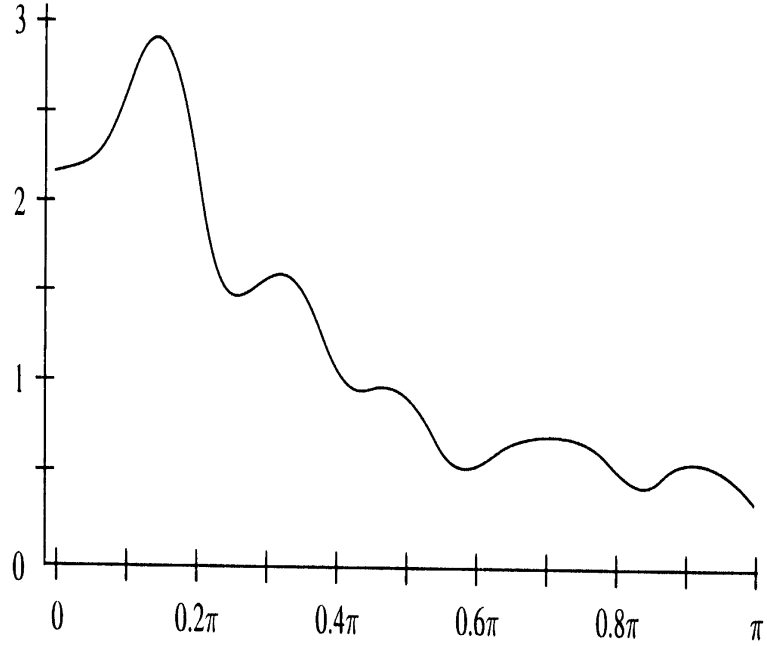
$$X^{(k)}(e^{\jmath\omega}) = \sum_{n=0}^{N-1} x^{(k)}[n]e^{-\jmath\omega n}\,; \qquad \widehat{S}_B(e^{\jmath\omega}) = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{N}\left|X^{(k)}(e^{\jmath\omega})\right|^2$$

REDUCING VARIANCE BY AVERAGING

ONE 16-POINT SEGMENT



32 16-POINT SEGMENTS



BLACKMAN-TUKEY METHOD

$$\widehat{S}_{BT}(e^{\jmath\omega}) = \sum_{l=-L}^{L} \mathsf{w}[l]\widehat{R}_x[l]e^{-\jmath\omega} \;\Leftrightarrow\; \widehat{S}_{BT}(e^{\jmath\omega}) = \frac{1}{2\pi}\mathsf{W}\,(e^{\jmath\omega}) \circledast \widehat{P}_x(e^{\jmath\omega})$$

MEAN

$$\mathcal{E}\left\{\widehat{S}_{BT}(e^{j\omega})\right\} = \frac{1}{4\pi^2}\mathsf{W}\;(e^{\jmath\omega}) \circledast \mathsf{W}_B\;(e^{\jmath\omega}) \circledast S_x(e^{\jmath\omega})$$

VARIANCE

Reduced by the factor (approximately):

$$\frac{1}{N_s}\sum_{l=-L}^{L} \mathsf{w}^2[l] = \frac{1}{2\pi N_s}\int_{-\pi}^{\pi} |\mathsf{W}(e^{\jmath\omega})|^2\, d\omega$$

WELCH METHOD

1. Divide data into K segments of length N.

2. Form $\widehat{P}_x^{(k)\prime}(e^{\jmath\omega}) = \dfrac{1}{N}\left|\displaystyle\sum_{n=0}^{N-1} \mathsf{v}[n]x^{(k)}[n]e^{-\jmath\omega n}\right|^2$

   where $\dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} \mathsf{v}^2[n] = 1.$

3. $\widehat{S}_W(e^{\jmath\omega}) = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K} \widehat{P}_x^{(k)\prime}(e^{\jmath\omega}).$

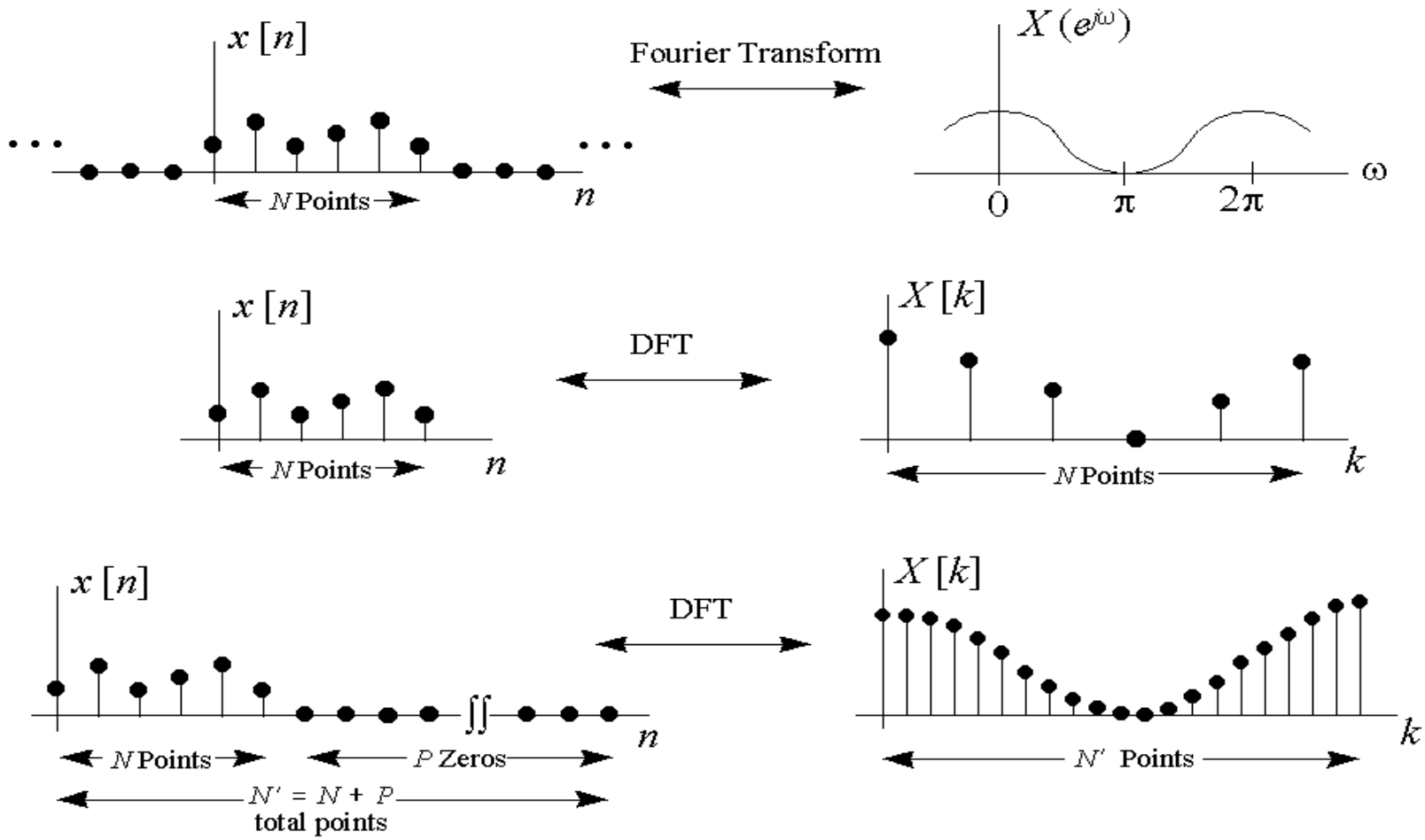
MEAN AND VARIANCE (WELCH METHOD)

MEAN

$$\mathcal{E}\left\{\widehat{S}_W(e^{\jmath\omega})\right\} = \frac{1}{2\pi} \cdot \frac{1}{N} \left|\mathsf{V}(e^{\jmath\omega})\right|^2 \circledast S_x(e^{\jmath\omega})$$

VARIANCE

For non-overlapping segments:

$$\mathrm{Var}\left[\widehat{S}_W(e^{\jmath\omega})\right] \approx \frac{1}{K} S_x^2(e^{\jmath\omega})$$

USE OF FFT FOR SPECTRAL ANALYSIS