Selected examples for Chapter 4.

EXAMPLE 4.101

The mean for a binary white noise process is

$$\mathrm{m}_x[n] = \mathcal{E}\{x[n]\} = 0$$

The correlation and covariance functions are given by

$$\begin{aligned} \mathrm{R}_x[n_1, n_0] = \mathrm{C}_x[n_1, n_0] &= \mathcal{E}\{x[n_1]x[n_0]\} \\ &= \begin{cases} 1 & \text{if } n_1 = n_0 \\ 0 & \text{if } n_1 \neq n_0 \end{cases} \end{aligned}$$

which can be written as

$$\mathrm{R}_x[n_1, n_0] = \mathrm{C}_x[n_1, n_0] = \delta[n_1 - n_0]$$

The process is wide-sense stationary because

1) $\mathrm{m}_x[n] = \text{constant}$

2) $\mathrm{R}_x[n_1, n_0]$ and $\mathrm{C}_x[n_1, n_0]$ are functions only of the *difference* $l = n_1 - n_0$

□

EXAMPLE 4.102

The mean for a general Bernoulli process is given by

$$\mathrm{m}_x[n] = \mathcal{E}\{x[n]\} = 2P - 1$$

The covariance function is given by

$$\begin{aligned}
\mathrm{C}_x[n_1, n_0] &= \mathcal{E}\{(x[n_1] - \mathrm{m}_x[n_1])(x[n_0] - \mathrm{m}_x[n_0])\} \\
&= \begin{cases} \mathrm{Var}\;[x[n_1]] = 4P(1-P) & \text{if } n_1 = n_0 \\ 0 & \text{if } n_1 \neq n_0 \end{cases}
\end{aligned}$$

This can be written as

$$\mathrm{C}_x[n_1, n_0] = 4P(1-P)\delta[n_1 - n_0]$$

The correlation function is thus given by

$$\begin{aligned}
\mathrm{R}_x[n_1, n_0] &= \mathrm{C}_x[n_1, n_0] + \mathrm{m}_x[n_1] \cdot \mathrm{m}_x[n_0] \\
&= 4P(1-P)\delta[n_1 - n_0] + (2P-1)^2
\end{aligned}$$

Since the mean is *constant* and R_x and C_x are a function of only $\ell = n_1 - n_0$, this process is wide-sense stationary. □

EXAMPLE 4.103

The mean for a discrete Wiener process is given by

$$\mathrm{m}_x[n] = \mathcal{E}\left\{x[n]\right\} = \mathcal{E}\left\{\sum_{i=1}^{n} \varepsilon[i]\right\} = \sum_{i=1}^{n} \mathcal{E}\left\{\varepsilon[i]\right\} = 0$$

The correlation and covariance functions can be found from

$$\begin{aligned}
\mathrm{R}_x[n_1, n_0] &= \mathrm{C}_x[n_1, n_0] = \mathcal{E}\left\{x[n_1]x[n_0]\right\} \\
&= \mathcal{E}\left\{\left(\sum_{k=1}^{n_1} \varepsilon[k]\right) \left(\sum_{i=1}^{n_0} \varepsilon[i]\right)\right\} \\
&= \sum_{k_1=1}^{\min\,(n_1, n_0)} \underbrace{\mathcal{E}\left\{\varepsilon^2[k]\right\}}_{1} = \min\,(n_1, n_0)
\end{aligned}$$

Since $\mathrm{R}_x[n_1, n_0]$ and $\mathrm{C}_x[n_1, n_0]$ are not a function of just the difference $\ell = n_1 - n_0$, this process is *not* wide-sense stationary. $\square$

EXAMPLE 4.1

Let $v[n]$ be a real-valued process of independent random variables each with mean μ and variance σ^2. The correlation function for this random process is

$$\mathrm{R}_v[n_1, n_0] = \mathcal{E}\{v[n_1]v[n_0]\} = \begin{cases} \mu^2 & n_1 \neq n_0 \\ \sigma^2 + \mu^2 & n_1 = n_0 \end{cases}$$

This can be written as

$$\mathrm{R}_v[n_1, n_0] = \sigma^2\delta[n_1 - n_0] + \mu^2$$

Since the mean is constant, and the correlation function is a function only of the difference $n_1 - n_0$, this random process is wide-sense stationary.

Now consider the process $x[n]$ defined by

$$x[n] = nv[n-1]$$

Its correlation function is

$$\begin{aligned} \mathrm{R}_x[n_1, n_0] &= \mathcal{E}\{x[n_1]x[n_0]\} = \mathcal{E}\{n_1 v[n_1 - 1] n_0 v[n_0 - 1]\} \\ &= n_1 n_0 \mathcal{E}\{v[n_1 - 1]v[n_0 - 1]\} = n_1 n_0 \left(\sigma^2\delta[n_1 - n_0] + \mu^2\right) \end{aligned}$$

Since this is *not* purely a function of $n_1 - n_0$, this random process is not wide-sense stationary. $\square$

EXAMPLE 4.2

Let $x[n]$ be a random process defined by

$$x[n] = v[n] + \frac{1}{2}v[n-1]$$

where $v[n]$ is the process defined in Example 4.1. Assume for simplicity that $\mu = 0$ and $\sigma^2 = 1$. The correlation function of this process is

$$\begin{aligned}
\mathrm{R}_x[n_1, n_0] &= \mathcal{E}\left\{x[n_1]x[n_0]\right\} \\
&= \mathcal{E}\left\{\left(v[n_1] + \frac{1}{2}v[n_1-1]\right)\left(v[n_0] + \frac{1}{2}v[n_0-1]\right)\right\} \\
&= \mathcal{E}\left\{v[n_1]v[n_0]\right\} + \frac{1}{2}\mathcal{E}\left\{v[n_1-1]v[n_0]\right\} + \frac{1}{2}\mathcal{E}\left\{v[n_1]v[n_0-1]\right\} \\
&\quad + \frac{1}{4}\mathcal{E}\left\{v[n_1-1]v[n_0-1]\right\} \\
&= \frac{5}{4}\delta[n_1 - n_0] + \frac{1}{2}\delta[n_1 - n_0 - 1] + \frac{1}{2}\delta[n_1 - n_0 + 1]
\end{aligned}$$

Since R_x is a function of just the difference $n_1 - n_0$, the random process is stationary. Further, it is easy to see that since $\mu = 0$, the covariance function and the correlation function are identical. □

EXAMPLE 4.3

Consider the complex random process defined by

$$x[n] = x_{\mathrm{r}}[n] + \jmath x_{\mathrm{i}}[n]$$

where $x_{\mathrm{r}}[n]$ and $x_{\mathrm{i}}[n]$ are *real* stationary random processes with mean zero and autocorrelation functions

$$R_{x_{\mathrm{r}}}[l] = R_{x_{\mathrm{i}}}[l] = \sigma^2\delta[l]$$

It is further assumed that the components x_{r} and x_{i} are orthogonal, that is

$$\mathcal{E}\left\{x_{\mathrm{r}}[n_1]x_{\mathrm{i}}[n_0]\right\} = 0$$

for all values of n_1 and n_0.

Since the weighted sum of two stationary random processes is stationary, the correlation function can be computed using the definition for a stationary random process

$$
\begin{aligned}
R_x[l] &= \mathcal{E}\left\{x[n]x^*[n-l]\right\} \\
&= \mathcal{E}\left\{(x_{\mathrm{r}}[n] + \jmath x_{\mathrm{i}}[n])(x_{\mathrm{r}}[n-l] - \jmath x_{\mathrm{i}}[n-l])\right\} \\
&= \mathcal{E}\left\{x_{\mathrm{r}}[n]x_{\mathrm{r}}[n-l]\right\} + \mathcal{E}\left\{x_{\mathrm{i}}[n]x_{\mathrm{i}}[n-l]\right\} - \jmath\mathcal{E}\left\{x_{\mathrm{r}}[n]x_{\mathrm{i}}[n-l]\right\} \\
&\qquad + \jmath\mathcal{E}\left\{x_{\mathrm{i}}[n]x_{\mathrm{r}}[n-l]\right\} \\
&= 2\sigma^2\delta[l]
\end{aligned}
$$

Since the means are zero, the covariance function is identical. $\square$

EXAMPLE 4.4

The correlation function for a certain random process has the exponential form

$$R_x[l] = 4(-0.5)^{|l|}$$

The correlation matrix for $N = 3$ is

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

which is clearly Toeplitz. The eigenvalues of this matrix are found to be $\lambda_1 = 7.4$, $\lambda_2 = 3.0$, and $\lambda_3 = 1.6$. Since the eigenvalues are all positive, the correlation matrix is *positive definite*.

(continued on the next page)

For $N = 4$ the correlation matrix has the Toeplitz form

$$\begin{bmatrix} 4 & -2 & 1 & -0.5 \\ -2 & 4 & -2 & 1 \\ 1 & -2 & 4 & -2 \\ -0.5 & 1 & -2 & 4 \end{bmatrix}$$

The eigenvalues of this matrix turn out to be $\lambda_1 = 8.3$, $\lambda_2 = 4.0$, $\lambda_3 = 2.2$, and $\lambda_4 = 1.5$ which implies that the matrix is positive definite, as required.

It can be shown by direct substitution that the correlation *function* given above satisfies the positive semidefinite condition with strict inequality (see Prob. 4.4 of text). This implies that the correlation matrix of *any* order is positive definite. □

EXAMPLE 4.5

A real random process has the exponential correlation function

$$R_x[l] = \sigma^2 \rho^{|l|}$$

The complex spectral density function for the process can be computed by

$$S_x(z) = \sum_{l=-\infty}^{\infty} \sigma^2 \rho^{|l|} z^{-l} = \sum_{l=0}^{\infty} \sigma^2 \rho^{l} z^{-l} + \sum_{l=-\infty}^{-1} \sigma^2 \rho^{-l} z^{-l}$$

The first term can be put in closed form by using the formula for an infinite geometric series, namely

$$\sum_{l=0}^{\infty} \sigma^2 \rho^l z^{-l} = \sigma^2 \sum_{l=0}^{\infty} (\rho z^{-1})^l = \frac{\sigma^2}{1-\rho z^{-1}}; \qquad |z| > |\rho|$$

(The condition for convergence of the series is $|\rho z^{-1}| < 1$ or $|z| > |\rho|$.)

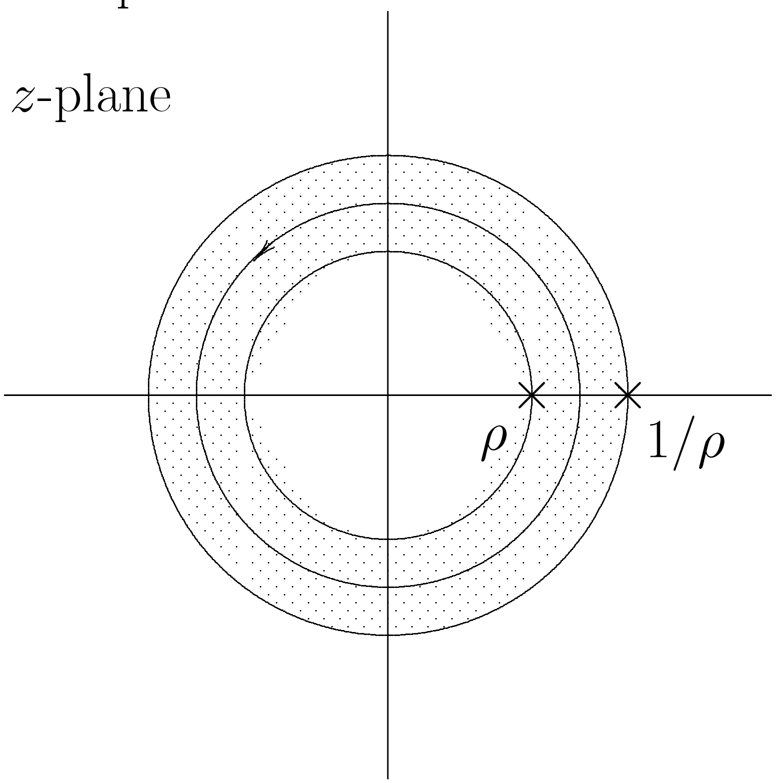
The second term can be written as

$$\sum_{k=1}^{\infty} \sigma^2 \rho^k z^k = \sigma^2 \rho z \sum_{l=0}^{\infty} (\rho z)^l = \frac{\sigma^2 \rho z}{1-\rho z}; \qquad |z| < 1/|\rho|$$

Therefore the complex power density spectrum is

$$\begin{aligned} S_x(z) &= \frac{\sigma^2}{1-\rho z^{-1}} + \frac{\sigma^2 \rho z}{1-\rho z} \\ &= \frac{\sigma^2(1-\rho^2)}{(1-\rho z^{-1})(1-\rho z)}; \qquad |\rho| < |z| < 1/|\rho| \end{aligned}$$

The poles of this function and the region of convergence are shown below.



$|\rho| < |z| < 1/|\rho|$

Clearly this z-transform exists only if $|\rho| < 1$.

The correlation function is recovered by integrating over a contour in the region of convergence such as the one shown. The integral is evaluated using residues according to the formula

$$
\begin{aligned}
R_x[l] = \frac{1}{2\pi \jmath} \oint S_x(z) z^{l-1} dz &= \sum \text{Residues } \left[S_x(z) z^{l-1} \right] \\
&= \sum \text{Residues } \left[\frac{\sigma^2(1-\rho^2) z^l}{(z-\rho)(1-\rho z)} \right]
\end{aligned}
$$

Note that for $l \geq 0$ there are no poles at the origin and the only pole enclosed by the contour is the one at $z = \rho$. Therefore the inverse transform for $l \geq 0$ is

$$
R_x[l] = \text{Res} \left[\frac{\sigma^2(1-\rho^2) z^l}{(z-\rho)(1-\rho z)} \text{ at } z = \rho \right] \tag{1}
$$

$$
= \frac{\sigma^2(1-\rho^2) z^l}{(z-\rho)(1-\rho z)} \cdot (z-\rho) \Big|_{z=\rho} = \sigma^2 \rho^l \tag{2}
$$

Since a real correlation function is known to be an even function of l, there is no actual need to carry out the inversion for $l < 0$. However to show how we could proceed, it is best to make the transformation of variables $z = 1/w$ and write the inversion formula as

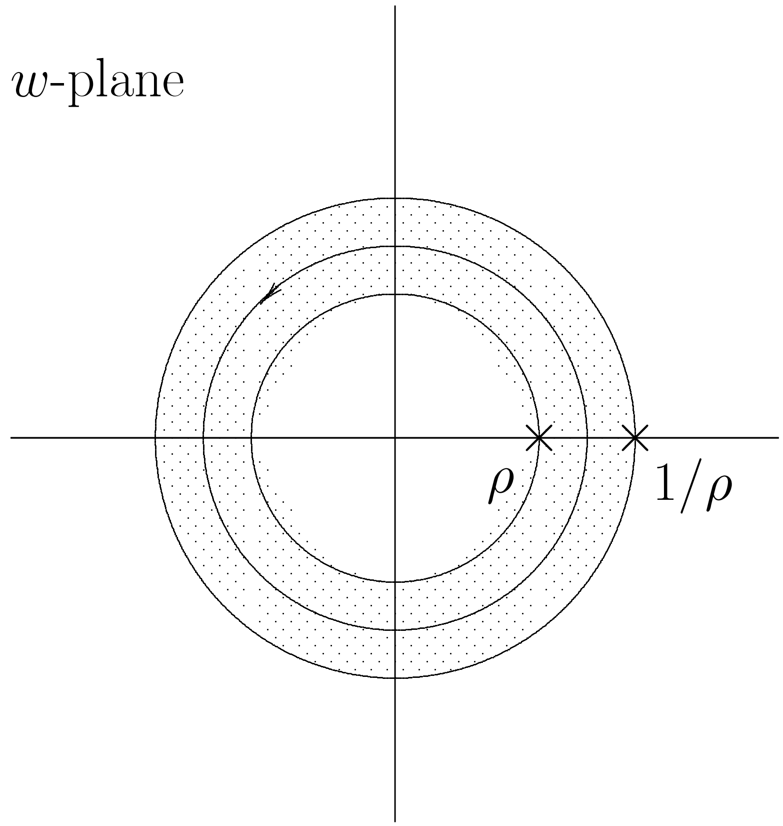
$$R_x[l] = \frac{1}{2\pi\jmath} \oint S_x(w^{-1})w^{-l-1}dw$$

The function

$$S_x(w^{-1})w^{-l-1} = \frac{\sigma^2(1-\rho^2)w^{-l}}{(w-\rho)(1-\rho w)}$$

has poles only at $w = \rho$ and $w = 1/\rho$ for $l < 0$ and converges in the annular region in between as shown below:

(continued on the next page)



$$|\rho| < |w| < 1/|\rho|$$

Ex.4.5(5)

The integration in the w plane is thus similar to the previous integral in the z plane and yields

$$\begin{aligned} R_x[l] &= \operatorname{Res}\left[\frac{\sigma^2(1-\rho^2)w^{-l}}{(w-\rho)(1-\rho w)} \text{ at } w=\rho\right] \\ &= \left.\frac{\sigma^2(1-\rho^2)w^{-l}}{(w-\rho)(1-\rho w)}\cdot(w-\rho)\right|_{w=\rho} = \sigma^2\rho^{-l} \end{aligned}$$

for $l<0$. The complete autocorrelation function is then

$$R_x[l] = \sigma^2\rho^{|l|} \qquad -\infty < l < \infty$$

$\square$

EXAMPLE 4.6

Suppose that the value of ρ in the previous example is 0.8 and the value of σ^2 is 2. Then the complex spectral density function is

$$S_x(z) = \frac{0.72}{(1-0.8z^{-1})(1-0.8z)} = \frac{-0.90z^{-1}}{(1-0.8z^{-1})(1-1.25z^{-1})}$$

The function can be expanded by partial fractions to obtain

$$S_x(z) = \frac{2}{1-0.8z^{-1}} - \frac{2}{1-1.25z^{-1}}$$

Now by noting the poles and the region of convergence (see Example 4.5) it can be seen that the first term corresponds to a '*causal*' sequence:

$$\frac{2}{1-0.8z^{-1}} \Leftrightarrow 2\cdot(0.8)^l \quad l \geq 0$$

The second term corresponds to an '*anticausal*' sequence:

$$\frac{-2}{1-1.25z^{-1}} \Leftrightarrow 2\cdot(1.25)^l = 2\cdot(0.8)^{-l} \quad l < 0$$

Putting these results together yields

$$R_x[l] = 2\cdot(0.8)^{|l|} \qquad -\infty < l < \infty \qquad \Box$$

EXAMPLE 4.104

A complex random process has the form $x[n] = A\,e^{j\omega_0 n}$ where $A = |A|e^{j\phi}$. The phase ϕ is uniformly distributed over $[-\pi, \pi]$ which implies that the complex amplitude A has zero mean.

The average power in the signal is

$$\mathsf{P}_{\mathrm{o}} = \mathcal{E}\left\{|x[n]|^2\right\} = \mathcal{E}\left\{x[n]x^*[n]\right\} = \mathcal{E}\left\{AA^*\right\} = \sigma_A^2$$

The correlation function is given by

$$\mathrm{R}_x[n_1, n_0] = \mathcal{E}\left\{Ae^{j\omega_0 n_1}A^*e^{-j\omega_0 n_0}\right\} = \sigma_A^2 e^{j\omega_0(n_1-n_0)}$$

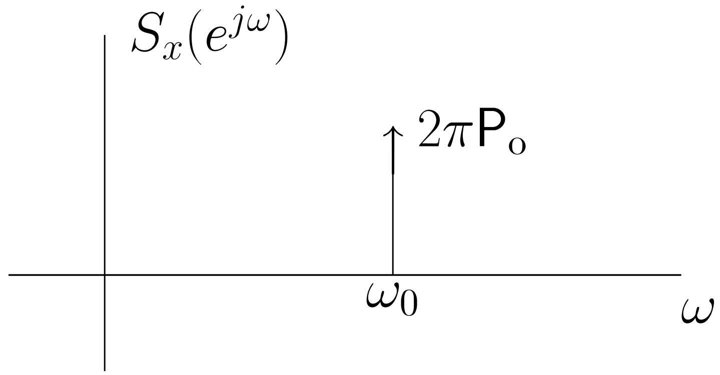
The dependency on $n_1 - n_0$ shows that the process is wide sense stationary, and since $\mathsf{P}_{\mathrm{o}} = \sigma_A^2$ we can write

$$R_x[\ell] = \mathsf{P}_{\mathrm{o}} e^{j\omega_0 \ell}$$

Since the power spectral density function is the Fourier transform of $R_x[\ell]$, we have

$$S_x(e^{j\omega}) = 2\pi\mathsf{P}_{\mathrm{o}}\delta_c\left(e^{j\omega} - e^{j\omega_0}\right)$$

$$S_x(e^{j\omega}) = 2\pi \mathsf{P}_\mathrm{o} \delta_c \left(e^{j\omega} - e^{j\omega_0}\right)$$



<u>Special Case</u>

If $|A|$ has a *Rayleigh* density

$$f_{|A|}(r) = \frac{1}{\sigma_\mathrm{o}^2} r\, e^{-r^2/2\sigma_\mathrm{o}^2}$$

Then the real and imaginary parts of $x[n]$ are independent zero-mean *Gaussian* random variables with variance $\sigma_\mathrm{o}^2 = \dfrac{2\mathsf{P}_\mathrm{o}}{4-\pi}$. □