COMPLEX FUNCTIONS AND THE BILATERAL Z-TRANSFORM

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The z-transform is an essential tool for the analysis of stochastic as well as deterministic signals and systems. Our intent here is to first provide just enough of the theory of complex functions to give the reader a clear understanding of the mathematical basis of the transform. With this as background, we proceed to define the z-transform, discuss its region of convergence, and its inversion by integration in the complex plane.

1 Review of Calculus for Complex Variables

In this section we review areas of complex variable calculus that are pertinent to the study of the z-transform. The intent here is to summarize the important results in a way that appeals to intuition. Absolutely no attempt is made to be mathematically rigorous and some results are presented without proof. Several very readable texts [1, 2, 3, 4] are listed in the references that develop the results in more detail and provide formal proofs.

1.1 Analytic Functions

Let z denote a complex variable. A complex-valued function $g(z)$ is said to be *analytic* at a point z_{o} if its derivative exists for every point in a neighborhood around z_{o}. The derivative of a function of a complex variable is defined as

$$g'(z) = \frac{dg(z)}{dz} = \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \tag{1}$$

The definition of the derivative looks exactly like that for a function of a real variable. However there is a significant difference. In order for the derivative of a real function to exist, one must have the same result regardless if the limit is approached from the right or the left (i.e. Δz is positive or negative). Likewise for the derivative of a complex function to exist the limit (1) must be the same regardless of the direction of Δz. However for a complex function Δz can be taken in an *infinite* number of possible directions (see Fig. 1). Therefore the conditions for existence of

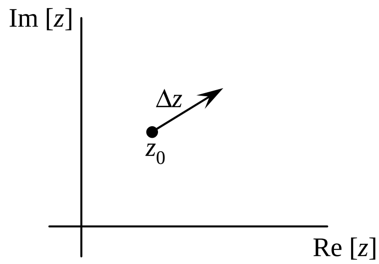

Figure 1: Displacement for defining the derivative at a point z_o.

the derivative for a complex function are much more stringent than those for the existence of the derivative of a real function.

Fortunately a simple set of conditions exists to test if the derivative can be defined at a given point z_o. These are the *Cauchy-Riemann* conditions. Let the complex variable z be represented as

$$z = z_r + \jmath z_i$$

and the function $g(\cdot)$ be represented by its real and imaginary parts

$$g(z) = g_r(z_r, z_i) + \jmath g_i(z_r, z_i)$$

Then the derivative of the complex function exists at a point z_o if the Cauchy-Riemann equations are satisfied:

$$\begin{aligned} \frac{\partial g_r(z_r, z_i)}{\partial z_r} &= \frac{\partial g_i(z_r, z_i)}{\partial z_i} \quad (a) \\ \frac{\partial g_r(z_r, z_i)}{\partial z_i} &= -\frac{\partial g_i(z_r, z_i)}{\partial z_r} \quad (b) \end{aligned} \tag{2}$$

(It is assumed that these partial derivatives exist and are continuous.)

It is easy to see that these conditions are necessary because if the displacement Δz is taken along the real axis ($\Delta z = \Delta z_r$) then[1]

$$\frac{dg(z)}{dz} = \frac{\partial g_r(z_r, z_i)}{\partial z_r} + \jmath \frac{\partial g_i(z_r, z_i)}{\partial z_r}$$

[1]The reader may want to express the derivatives as a limit as in (1) to see that the following two expressions hold.

while if the displacement is taken along the imaginary axis ($\Delta z = \jmath \Delta z_{\rm i}$) then

$$\frac{dg(z)}{dz} = -\jmath \frac{\partial g_{\rm r}(z_{\rm r}, z_{\rm i})}{\partial z_{\rm i}} + \frac{\partial g_{\rm i}(z_{\rm r}, z_{\rm i})}{\partial z_{\rm i}}$$

Since the two expressions for the derivative must be the same, the Cauchy-Riemann conditions (2) must hold. The sufficiency of the conditions is more difficult to show but nevertheless obtains [see e.g. [2]].

Frequently a function of a complex variable fails to be analytic at only one or more isolated points in the plane but remains analytic in at least small regions surrounding those points. Such points will be referred to as *isolated singularities* or *poles* of the function. It will be seen that these points play an important role in integration of functions of complex variables.

1.2 Series Expansions

If a function is analytic in a region around and including $z_{\rm o}$, then it can be expanded in a Taylor series

$$g(z) = g(z_{\rm o}) + g'(z_{\rm o})(z - z_{\rm o}) + \frac{1}{2!} g''(z_{\rm o})(z - z_{\rm o})^2 + \cdots + \frac{1}{n!} g^{(n)}(z_{\rm o})(z - z_{\rm o})^n + \cdots \tag{3}$$

If a function is not analytic at $z_{\rm o}$ but is analytic on and between two circular contours surrounding $z_{\rm o}$ (see Fig. 2), then it can be expanded in a *Laurent*

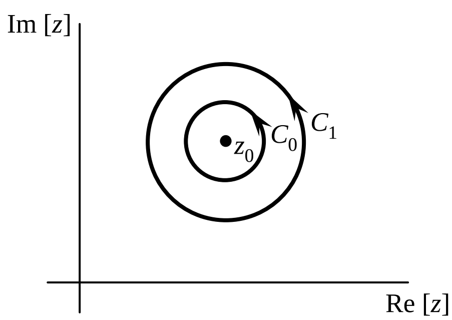

Figure 2: Region for expansion of a function in a Laurent series.

series

$$\begin{aligned} g(z) = \cdots + \frac{a_{-n}}{(z - z_{\rm o})^n} &+ \frac{a_{-n+1}}{(z - z_{\rm o})^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_{\rm o})} \\ &+ a_0 + a_1(z - z_{\rm o}) + a_2(z - z_{\rm o})^2 + \cdots \end{aligned} \tag{4}$$

where the upper and lower limits of the series may be finite or infinite. This series representation is fundamental to the properties of contour integration that follow.

1.3 Contour Integration

A function of a complex variable may be integrated along a contour (i.e. a curved line) in the complex plane. This integral is represented as

$$\int_C g(z)dz = \lim_{\Delta z_k \to 0} \sum_C g(z_k)\Delta z_k \tag{5}$$

where the line is assumed to be divided into a series of very small segments, z_k is a point on the segment, and Δz_k is a small vector connecting the end points of the k^{th} segment (see Fig. 3).

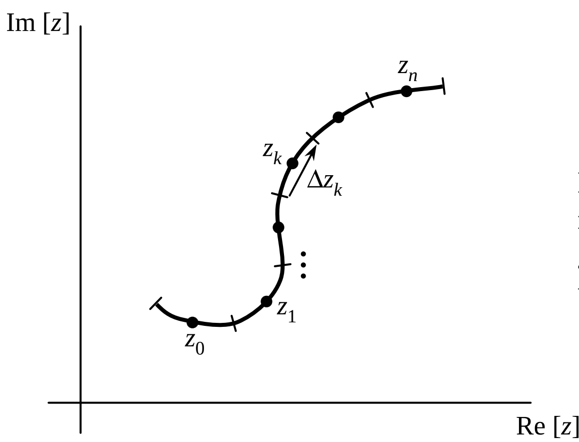

Figure 3: Small segments involved in integration along a line in the complex plane.

The contour integral can be written in terms of real and imaginary parts as

$$\begin{aligned}\int_C g(z)dz &= \int_C (g_\mathrm{r}(z_\mathrm{r}, z_\mathrm{i}) + \jmath g_\mathrm{i}(z_\mathrm{r}, z_\mathrm{i}))\,(dz_\mathrm{r} + \jmath dz_\mathrm{i}) \\ &= \int_C (g_\mathrm{r}(z_\mathrm{r}, z_\mathrm{i})dz_\mathrm{r} - g_\mathrm{i}(z_\mathrm{r}, z_\mathrm{i})dz_\mathrm{i}) \\ &\quad + \jmath \int_C (g_\mathrm{r}(z_\mathrm{r}, z_\mathrm{i})dz_\mathrm{i} + g_\mathrm{i}(z_\mathrm{r}, z_\mathrm{i})dz_\mathrm{r}) \end{aligned} \tag{6}$$

If the contour is *closed*, i.e. the beginning and end points are the same, the integral is written as

$$\oint g(z)dz = \oint (g_{\rm r}(z_{\rm r}, z_{\rm i})dz_{\rm r} - g_{\rm i}(z_{\rm r}, z_{\rm i})dz_{\rm i}) + \jmath \oint (g_{\rm i}(z_{\rm r}, z_{\rm i})dz_{\rm r} + g_{\rm r}(z_{\rm r}, z_{\rm i})dz_{\rm i}) \tag{7}$$

where by convention the direction of integration is taken to be counter-clockwise on the contour.

If $g(\cdot)$ is analytic everywhere on and inside the contour then we can show that

$$\oint g(z)dz = 0 \tag{8}$$

This result is known as the *Cauchy-Goursat Theorem*, and its implications are important for the material to follow. Cauchy originally stated and proved the result for functions with continuous partial derivatives. Goursat later showed that continuity of the partial derivatives was not necessary and gave an alternate proof [see e.g. [3]]. We will give a brief sketch of the proof here using partial derivatives. The essence of the proof is to first show that the the line integral over a closed contour can be expressed as an integral of the partial derivative over the enclosed area[2]. Once this is done, the Cauchy-Riemann conditions can be applied to show that the integral is in fact zero.

To begin, consider the first term on the right of (7),

$$\oint g_{\rm r}(z_{\rm r}, z_{\rm i})dz_{\rm r} \tag{9}$$

and consider a small interval along the $z_{\rm r}$ axis as shown in Fig. 4. The contribution to the integral along the lower and upper parts of the contour is

$$g_{\rm r}(z_{\rm r}, a)\Delta z_{\rm r} - g_{\rm r}(z_{\rm r}, b)\Delta z_{\rm r}$$

(The minus sign occurs because the direction of integration for the top segment is opposite to the direction of the $z_{\rm r}$ axis.) Therefore the contour integral (9) can be represented by the ordinary integral

$$\int (g_{\rm r}(z_{\rm r}, a) - g_{\rm r}(z_{\rm r}, b))\, dz_{\rm r}$$

where the values a and b are of course a function of $z_{\rm r}$. Now notice that we can write

[2]The result applied to real variables leads to an identity in advanced calculus known as *Green's Theorem* [see e.g. [5]].

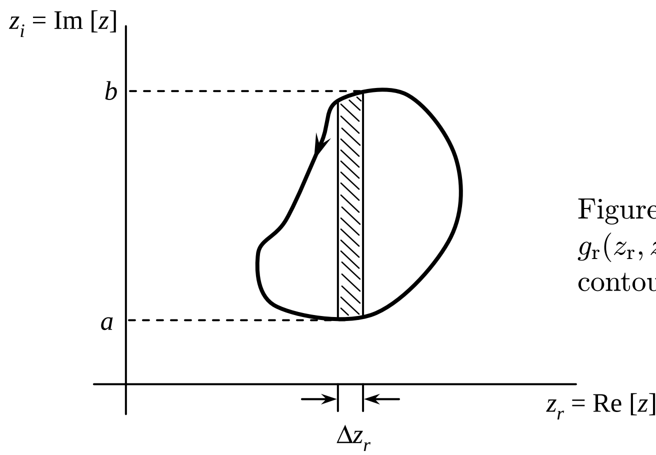

Figure 4: Integration of $g_{\mathrm{r}}(z_{\mathrm{r}}, z_{\mathrm{i}})$ along a closed contour.

$$g_{\mathrm{r}}(z_{\mathrm{r}}, a) - g_{\mathrm{r}}(z_{\mathrm{r}}, b) = -\int_a^b \left(\frac{\partial g_{\mathrm{r}}(z_{\mathrm{r}}, z_{\mathrm{i}})}{\partial z_{\mathrm{i}}}\right) dz_{\mathrm{i}}$$

where the integral is along the vertical strip shown in Fig. 4. Putting all of this together we find

$$\oint g_{\mathrm{r}}(z_{\mathrm{r}}, z_{\mathrm{i}}) dz_{\mathrm{r}} = -\int\int \left(\frac{\partial g_{\mathrm{r}}}{\partial z_{\mathrm{i}}}\right) dz_{\mathrm{i}} dz_{\mathrm{r}} \tag{10}$$

where the double integral on the right is over the area enclosed by the contour. Consideration of the other terms in (7) leads to similar expressions. The result is that the integral in (7) can be written as

$$\oint g(z) dz = -\int\int \left[\left(\frac{\partial g_{\mathrm{r}}}{\partial z_{\mathrm{i}}}\right) + \left(\frac{\partial g_{\mathrm{i}}}{\partial z_{\mathrm{r}}}\right)\right] dz_{\mathrm{i}} dz_{\mathrm{r}} + \jmath \int\int \left[\left(\frac{\partial g_{\mathrm{r}}}{\partial z_{\mathrm{r}}}\right) - \left(\frac{\partial g_{\mathrm{i}}}{\partial z_{\mathrm{i}}}\right)\right] dz_{\mathrm{i}} dz_{\mathrm{r}} \tag{11}$$

Since by the Cauchy-Riemann conditions (2) the integrands are zero, the result (8) is proven.

The Cauchy-Goursat theorem can be used to develop a simple expression for the contour integral in a region where there are isolated singularities (poles). For this, let n be any positive or negative integer and consider the integral

$$\oint (z - z_{\mathrm{o}})^n dz \tag{12}$$

for an arbitrary point z_o in the complex plane over a contour which is a circle of radius r (see Fig. 5). The function to be integrated can be shown

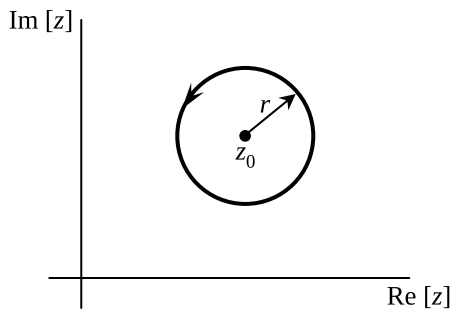

Figure 5: Integration of the function $1/(z-z_\mathrm{o})^{n+1}$ over a circular contour.

from the Cauchy-Reimann conditions to be analytic everywhere except at the point $z = z_\mathrm{o}$. Since the above integral is over a circle centered at z_o, it can be written as

$$\begin{aligned}
\oint (z - z_\mathrm{o})^n dz &= \int_0^{2\pi} \left(r^n e^{\jmath n\phi} \right) \jmath r e^{\jmath\phi} d\phi \\
&= \jmath r^{n+1} \int_0^{2\pi} e^{-\jmath(n+1)\phi} d\phi \\
&= \jmath r^{n+1} \int_0^{2\pi} \left(\cos(n+1)\phi - \jmath \sin(n+1)\phi\right) d\phi
\end{aligned} \tag{13}$$

Since the integration is over $n + 1$ complete periods of the cosine and sine, the integral is zero unless $n = -1$. In that case the cosine is one and the sine is zero so the integral is equal to 2π. Thus we have just shown the important result:

$$\oint_{|z-z_\mathrm{o}|=r} (z - z_\mathrm{o})^n dz = \left\{ \begin{array}{ll} 2\pi\jmath & n = -1 \\ 0 & \text{otherwise} \end{array} \right. \tag{14}$$

Now suppose that $g(z)$ is a function that is analytic everywhere in the circular region except at the point z_o, and consider its integral over the circular contour. By expanding $g(z)$ in a Laurent series and integrating term by term, we see from (4) and (14) that the only term that contributes to the integral is the term

$$\frac{a_{-1}}{(z - z_\mathrm{o})}$$

The value of the integral is therefore

$$\oint_{|z-z_o|=r} g(z)dz = 2\pi \jmath a_{-1} \tag{15}$$

The term a_{-1} is called the *residue* of the pole of the function $g(z)$ at z_o.

We now come to our final result for this section. Suppose that $g(z)$ is a complex function that is analytic everywhere on and inside a closed contour C except at a finite number of poles at $z_1, z_2, \ldots, z_p$. Then form the extended contour shown in Fig. 6 and note that the function $g(\cdot)$, by assumption, is

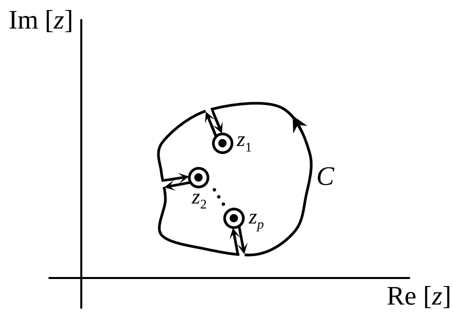

Figure 6: Integration of a function $g(z)$ over a contour surrounding some of its poles.

analytic everywhere on and inside of the extended contour. The Cauchy-Gourmat theorem then states that the integral on the extended contour is equal to zero. Note that if the function is continuous and the paths to and from the poles are taken sufficiently close together, their contributions to the integral will cancel. (This follows because the integration along these paths is taken in opposite directions.) Therefore the integral over the original outer contour C must be equal to the integral over all of the little circles surrounding the poles. The integral over each circle by (15) is in turn equal to $2\pi\jmath$ times the residue at the corresponding pole. It thus follows that

$$\oint g(z)dz = 2\pi\jmath \sum \text{Residues (at poles within the contour)} \tag{16}$$

Example 1 Consider the rational polynomial function

$$g(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z-z_1)(z-z_2)^2(z-z_3)}$$

Suppose this function is integrated over a contour that encloses poles z_1 and z_2 but not z_3. The integral is given by

$$\oint g(z)dz = 2\pi\jmath \left\{ \text{Res}\left[g(z) \text{ at } z=z_1\right] + \text{Res}\left[g(z) \text{ at } z=z_2\right] \right\}$$

(Since the contour does not enclose the pole at z_3 its residue is not needed.) To find the residue at z_1 first notice that the Laurent series for $g(z)$ must not have any negative powers beyond the $a_{-1}/(z-z_1)$ term. If there were any further negative powers such as $a_{-2}/(z-z_1)^2$ in the Laurent series then the function $g(z)\cdot(z-z_1)$ would still have a pole at $z=z_1$. However it is clear from the above definition of $g(z)$ that multiplying by $(z-z_1)$ *cancels* the pole at $z=z_1$ and so $g(z)\cdot(z-z_1)$ is analytic at $z=z_1$. The Laurent series for $g(z)$ must therefore start at the term $a_{-1}/(z-z_1)$ and has the form

$$g(z) = \frac{a_{-1}}{(z-z_1)} + a_0 + a_1(z-z_1) + a_2(z-z_2)^2 + \cdots$$

If this expression is multiplied by $(z-z_1)$ and the result is evaluated at $z=z_1$, all that is left is the residue a_{-1}. That is,

$$\mathrm{Res}\,[g(z) \text{ at } z=z_1] = g(z)(z-z_1)\big|_{z=z_1}$$

The explicit result for this example is

$$\mathrm{Res}\,[g(z) \text{ at } z=z_1] = \frac{P(z_1)}{(z_1-z_2)^2(z_1-z_3)}$$

To find the residue at z_2 observe that the Laurent series there cannot have any negative powers beyond the $a_{-2}/(z-z_2)^2$ term. Again, if it did, the function $g(z)(z-z_2)^2$ would not be analytic at $z=z_2$. Thus $g(z)$ can be expanded in a Laurent series about $z=z_2$ as

$$g(z) = \frac{a_{-2}}{(z-z_2)^2} + \frac{a_{-1}}{(z-z_2)} + a_0 + a_1(z-z_2) + \cdots$$

To find a_{-1} one can multiply by $(z-z_2)^2$, which results in

$$f(z)(z-z_2)^2 = a_{-2} + a_{-1}(z-z_2) + a_0(z-z_2)^2 + \cdots$$

then take the derivative and evaluate the result at z_2. In other words,

$$\mathrm{Res}\,[g(z) \text{ at } z=z_2] = \frac{d}{dz}\left[g(z)(z-z_2)^2\right]\Big|_{z=z_2}$$

The explicit result for this example is

$$\mathrm{Res}\,[g(z) \text{ at } z=z_2] = \frac{(z_2-z_1)(z_2-z_3)P'(z_2) - P(z_2)(2z_2-z_1-z_3)}{(z_2-z_1)^2(z_2-z_3)^2}$$

The integral of this function $g(z)$ over the closed contour is then $2\pi\jmath$ times the sum of the two residues.

□

The example shows how to evaluate the residue of a function at a first and second order pole. In general, for a function with a k^{th} order pole at $z = z_o$, the residue is given by

$$\text{Res}\,[g(z) \text{ at } z = z_o] = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[g(z)(z-z_o)^k\right]\bigg|_{z=z_o} \tag{17}$$

The factor $1/(k-1)!$ is needed to compensate for the terms that result in taking the derivative $k-1$ times.

2 The z-Transform and its Inverse

Given a sequence $\{x(n)\}$, we define the z-transform of the sequence as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{18}$$

Consider now the convergence of this sum. The z-transform can be written as

$$X(z) = X_+(z) + X_-(z) \tag{19}$$

where

$$X_+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{20}$$

and

$$X_-(z) = \sum_{n=-\infty}^{-1} x(n)z^{-n} = \sum_{k=1}^{\infty} x(-k)z^{k} \tag{21}$$

Consider first the convergence of $X_+(z)$. The series can be shown to converge uniformly (i.e., for every value of z) in a region outside of some circle if it converges absolutely [2]; that is if

$$\sum_{n=0}^{\infty} |x(n)||z|^{-n} < \infty \tag{22}$$

Now suppose that the right-sided series $X_+(z)$ converges absolutely at some point $z = z_1$. Then it also converges at any value of z outside of a circle defined by $|z| > |z_1|$ since

$$\sum_{n=0}^{\infty} |x(n)||z|^{-n} < \sum_{n=0}^{\infty} |x(n)||z_1|^{-n} < \infty$$

Let the smallest such circle have a radius r_R. Then $X_+(z)$ converges uniformly everywhere *outside* of a circle defined by

$$|z| > r_R$$

In a similar fashion the series $X_-(z)$, which involves only positive powers of z, can be seen to converge everywhere *inside* of a circle of some radius r_L. As a result, the z-transform of a sequence will converge in general in an annular region defined by

$$r_R < |z| < r_L \tag{23}$$

and is analytic there. More specific forms for the region of convergence obtain if the sequence has specific properties such as finite length or right-sidedness [6].

To find the inverse z-transform it is only necessary to note that by its definition the z-transform is a Laurent series about the point $z = 0$. Therefore if the function $X(z)$ is integrated over a closed contour surrounding the origin in the region of convergence the result is

$$\oint X(z)dz = 2\pi \jmath\, x(1) \tag{24}$$

(This follows from (18) since $x(1)$ is the residue or the coefficient of the z^{-1} term.) Likewise if the modified function $X(z)z^{n-1}$ is integrated over this same closed contour, the result is $2\pi\jmath$ times the residue of

$$X(z)z^{n-1} = \sum_{k=-\infty}^{\infty} x(k)z^{n-k-1}$$

or $2\pi\jmath\, x(n)$. Thus the inverse z-transform is given by

$$x(n) = \frac{1}{2\pi\jmath} \oint X(z)z^{n-1}dz \tag{25}$$

where the integration is over a closed contour surrounding the origin in the region of convergence. The integration is carried out as described in the previous section by evaluating residues of the poles enclosed by the contour.

The following two examples show how to compute the z-transform, its region of convergence, and the inverse z-transform for some relatively simple sequences. Since contour integration can become difficult for more general types of sequences, other methods such as partial fraction expansion are frequently used to recover a sequence from its z-transform. However, even in the use of these other methods an understanding of region of convergence and the principles of integration in the complex plane are valuable assets for the analysis of signals, systems, and power spectral density using the z-transform.

Example 2 This example is adapted from [6]. A causal linear system has a unit sample response of the form

$$h(n) = \begin{cases} \alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

The z-transform is given by

$$\begin{aligned} H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} &= \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}; \qquad |z| > |\alpha| \end{aligned}$$

The condition for convergence of the series, $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$ defines the region of convergence. The pole-zero plot for this function, and the region of convergence are shown in Fig. 7.

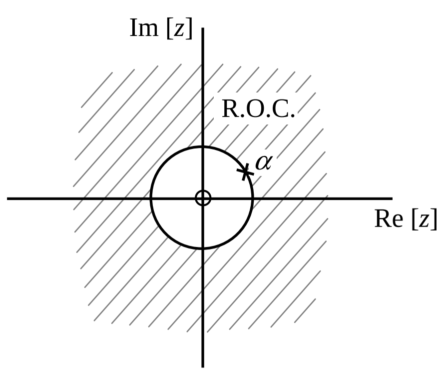

Figure 7: Pole and zero and region of convergence for system function $H(z) = z/(z-\alpha)$.

To recover the sequence from the z-transform we use (25)

$$h(n) = \frac{1}{2\pi \jmath} \oint H(z) z^{n-1} dz = \sum \text{Residues} \left[\frac{z^n}{z-\alpha} \right]$$

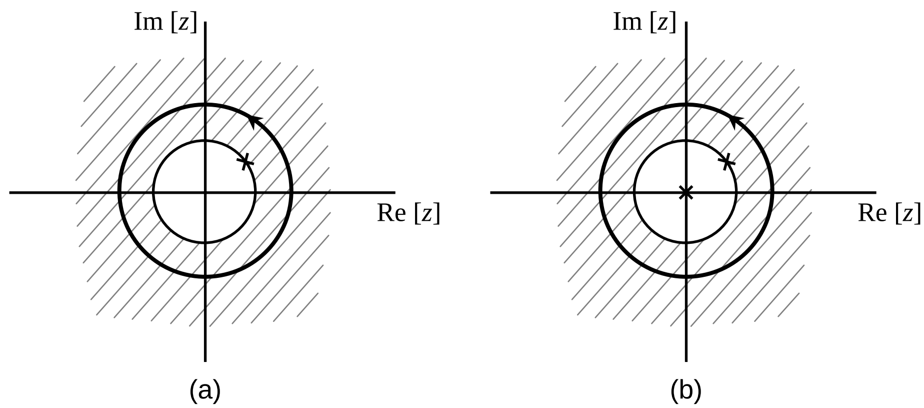

Figure 8: Evaluation of contour integral of $z^n/(z-\alpha)$. (a) $n \geq 0$. (b) $n < 0$.

The contour of integration is shown in Fig. 8(a). Notice that for $n \geq 0$ there is only a single pole at $z = \alpha$ and so

$$h(n) = \text{Res}\left[\frac{z^n}{(z-\alpha)} \text{ at } z = \alpha\right] = \left.\frac{z^n}{(z-\alpha)} \cdot (z-\alpha)\right|_{z=\alpha} = \alpha^n$$

For $n < 0$ we have the same pole at $z = \alpha$ and the residue is evaluated as above. However we also have an $|n|^{th}$ order pole at $z = 0$ (see Fig. 8(b)). For $n = -1$ we have

$$\text{Res}\left[\frac{1}{z(z-\alpha)} \text{ at } z = \alpha\right] = \left.\frac{1}{z(z-\alpha)} \cdot (z-\alpha)\right|_{z=\alpha} = \frac{1}{\alpha}$$

and

$$\text{Res}\left[\frac{1}{z(z-\alpha)} \text{ at } z = 0\right] = \left.\frac{1}{z(z-\alpha)} \cdot z\right|_{z=0} = -\frac{1}{\alpha}$$

Thus the sum of the residues is zero and $h(-1) = 0$.
For $n = -2$ we have

$$\text{Res}\left[\frac{1}{z^2(z-\alpha)} \text{ at } z = \alpha\right] = \left.\frac{1}{z^2(z-\alpha)} \cdot (z-\alpha)\right|_{z=\alpha} = \frac{1}{\alpha^2}$$

and

$$\text{Res}\left[\frac{1}{z^2(z-\alpha)} \text{ at } z = 0\right] = \left.\frac{d}{dz}\left(\frac{1}{z^2(z-\alpha)} \cdot z^2\right)\right|_{z=0} = \left.-\frac{1}{(z-\alpha)^2}\right|_{z=0} = -\frac{1}{\alpha^2}$$

Again the sum of the residues is zero and $h(-2) = 0$. For larger negative values of n we have

$$\text{Res}\left[\frac{1}{z^{-n}(z-\alpha)} \text{ at } z = \alpha\right] = \left.\frac{1}{z^{-n}(z-\alpha)} \cdot (z-\alpha)\right|_{z=\alpha} = \frac{1}{\alpha^{-n}}$$

and with some difficulty we find

$$\begin{aligned}\text{Res}\left[\frac{1}{z^{-n}(z-\alpha)} \text{ at } z=0\right] &= \frac{1}{(-n-1)!}\frac{d^{-n-1}}{dz^{-n-1}}\left(\frac{1}{z^{-n}(z-\alpha)}\cdot z^{-n}\right)\Bigg|_{z=0} \\ &= \frac{(-1)^{-n-1}}{(-n-1)!}\frac{(-n-1)!}{(z-\alpha)^{-n}}\Bigg|_{z=0} = -\frac{1}{\alpha^{-n}}\end{aligned}$$

Since the two residues again sum to zero we find that in general $h(n) = 0$ for $n < 0$.

The inversion of the z-transform for $n < 0$ using (25) directly is seen to be rather tedious. A much easier procedure is to consider the mapping $w = 1/z$. For this we have

$$z = \frac{1}{w} \quad \text{and} \quad dz = -\frac{1}{w^2}dw$$

If these substitutions are made in (25) we find that

$$h(n) = \frac{1}{2\pi \jmath}\oint_{\text{CW}} H(1/w)\left(\frac{1}{w}\right)^{n-1}\left(-\frac{1}{w^2}dw\right) = \frac{1}{2\pi \jmath}\oint H(w^{-1})w^{-n-1}dw$$

where the notation 'cw' on the first integral indicates that the contour of integration in the w plane is originally in a clockwise direction due to the mapping $w = 1/z$. In the final expression the contour is taken in the usual counterclockwise direction. This eliminates the minus sign.

The function to be integrated is

$$H(w^{-1})w^{-n-1} = \frac{w^{-1}}{w^{-1}-\alpha}w^{-n-1} = \frac{(1/\alpha)w^{-n-1}}{w-(1/\alpha)}$$

For $n \le -1$ there are no poles at $w = 0$ and the region of convergence is $|w| < |1/\alpha|$. The function and a contour of integration are depicted in Fig. 9. Since there are no poles within the contour of integration

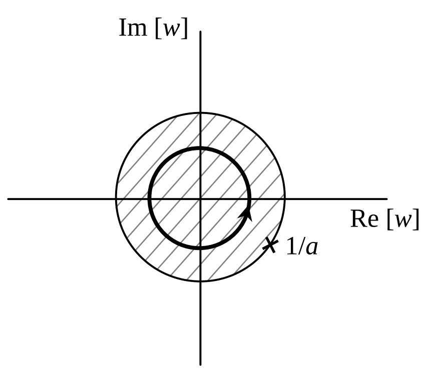

Figure 9: Evaluation of the integral of $H(1/w)w^{-n-1}$.

we find

$$h(n) = \frac{1}{2\pi \jmath} \oint H(w^{-1}) w^{-n-1} dw = 0$$

for $n < 0$.

$\square$

Example 3 An autocorrelation function has the form

$$R_x(l) = \sigma^2 \rho^{|l|}$$

The complex spectral density function is given by

$$S_x(z) = \sum_{l=-\infty}^{\infty} \sigma^2 \rho^{|l|} z^{-l} = \sum_{l=0}^{\infty} \sigma^2 \rho^l z^{-l} + \sum_{l=-\infty}^{-1} \sigma^2 \rho^{-l} z^{-l}$$

The first term yields an expression analogous to the one in the previous example, namely

$$\frac{\sigma^2 z}{(z-\rho)}; \qquad |z| > |\rho|$$

The second term can be written as

$$\sum_{k=1}^{\infty} \sigma^2 \rho^k z^k = \sigma^2 \rho z \sum_{k=1}^{\infty} (\rho z)^{k-1} = \frac{\sigma^2 \rho z}{(1-\rho z)}; \qquad |z| < 1/|\rho|$$

Therefore the complex power density spectrum is

$$\begin{aligned} S_x(z) &= \frac{\sigma^2 z}{(z-\rho)} + \frac{\sigma^2 \rho z}{(1-\rho z)} \\ &= \frac{\sigma^2 (1-\rho^2) z}{(z-\rho)(1-\rho z)}; \qquad |\rho| < |z| < 1/|\rho| \end{aligned}$$

The zero and poles of this function and the region of convergence are depicted in Fig. 10. Clearly this z-transform exists only if $|\rho| < 1$.

The inverse z-transform is computed from the relation

$$R_x(l) = \frac{1}{2\pi \jmath} \oint S_x(z) z^{l-1} dz = \sum \text{Residues} \left[\frac{\sigma^2 (1-\rho^2) z^l}{(z-\rho)(1-\rho z)} \right]$$

Fig. 11 shows the contour of integration in the complex plane. Note that for $l \geq 0$ there are no poles at the origin and the only pole enclosed

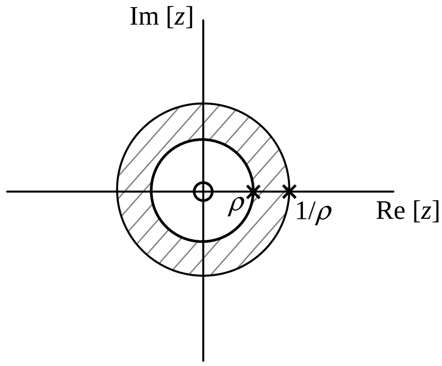

Figure 10: Pole-zero plot of a complex spectral density function showing the region of convergence.

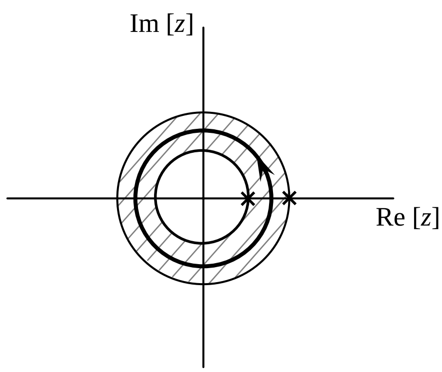

Figure 11: Contour of integration for complex spectral density function.

by the contour is the one at $z = \rho$. Therefore the inverse transform for $l \geq 0$ is

$$\begin{aligned} R_x(l) &= \operatorname{Res}\left[\frac{\sigma^2(1-\rho^2)z^l}{(z-\rho)(1-\rho z)} \text{ at } z = \rho\right] \\ &= \left.\frac{\sigma^2(1-\rho^2)z^l}{(z-\rho)(1-\rho z)} \cdot (z-\rho)\right|_{z=\rho} = \sigma^2 \rho^l \end{aligned}$$

Since a real correlation function is known to be an even function of l there is no actual need to carry out the inversion for $l < 0$. However to show how one could proceed, it is best to make the transformation of variables $z = 1/w$ and write the inversion formula as

$$R_x(l) = \frac{1}{2\pi \jmath} \oint S_x(w^{-1}) w^{-l-1} dw$$

The function

$$S_x(w^{-1}) w^{-l-1} = \frac{\sigma^2(1-\rho^2)w^{-l}}{(w-\rho)(1-\rho w)}$$

has poles only at $w = \rho$ and $w = 1/\rho$ for $l < 0$ and converges in the annular region in between (see Fig. 12). The integration in the w plane

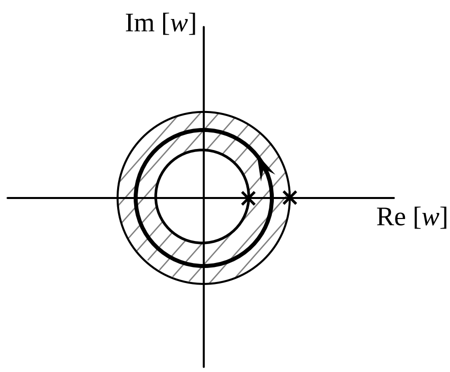

Figure 12: Contour of integration for complex spectral density function in the w plane.

is thus similar to the previous integral in the z plane and yields

$$\begin{aligned} R_x(l) &= \operatorname{Res}\left[\frac{\sigma^2(1-\rho^2)w^{-l}}{(w-\rho)(1-\rho w)} \text{ at } w = \rho\right] \\ &= \left.\frac{\sigma^2(1-\rho^2)w^{-l}}{(w-\rho)(1-\rho w)} \cdot (w-\rho)\right|_{w=\rho} = \sigma^2 \rho^{-l} \end{aligned}$$

for $l < 0$. The complete autocorrelation function is then

$$R_x(l) = \sigma^2 \rho^{|l|}$$

□

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