EXTRA CREDIT

1.

$$F_X(x) = (1 - e^{-x/5})\, u(x)$$

a. $P[5 \le X \le 7] = F_X(7) - F_X(5)$

$= (1 - e^{-7/5}) - (1 - e^{-5/5})$

$= e^{-1} - e^{-7/5}$

$= \underline{0.1213}$

b. $P[X \ge 3] = 1 - P[X < 3] = 1 - F_X(3)$

$= 1 - (1 - e^{-3/5}) = e^{-3/5}$

$= \underline{0.5488}$

c. $P[X < 3] = F_X(3) = \underline{0.4512}$

d. $f_X(x) = \frac{d F_X(x)}{dx} = -(-\frac{1}{5}) e^{-x/5} u(x) = \underline{\frac{1}{5} e^{-x/5} u(x)}$

2. $Y = X^3 \quad f_X(x) = \frac{1}{3} \quad 2 \le X \le 5$

a. $E[Y] = E[X^3] = \int_2^5 x^3 \frac{1}{3}\, dx = \frac{x^4}{12}\Big|_2^5 = \frac{5^4 - 2^4}{12} = \underline{50.75}$

$E[Y^2] = E[X^6] = \int_2^5 x^6 \frac{1}{3}\, dx = \frac{x^7}{21}\Big|_2^5 = \frac{5^7 - 2^7}{21} = \underline{3714.14}$

$\sigma_Y^2 = E[Y^2] - E[Y]^2 = 3714.14 - 50.75^2 = \underline{1138.58}$

b. $\Phi_X(\omega) = \sum_{k=0}^{\infty} \frac{E[X^k]}{k!}(j\omega)^k$ (Maclaurin)

$$= 1 + E[X]j\omega + E[X^2]\frac{(j\omega)^2}{2} + \cdots$$

$$= e^{\alpha(e^{j\omega}-1)}$$

$$= e^{\alpha\left(\sum_{k=0}^{\infty}\frac{(j\omega)^k}{k!}-1\right)} = \sum_{m=0}^{\infty}\frac{\left(\alpha\sum_{k=0}^{\infty}\frac{(j\omega)^k}{k!}-1\right)^m}{m!}$$

$$= 1+\alpha\left(\cancel{1}+j\omega+\frac{(j\omega)^2}{2}+\cdots-\cancel{1}\right)+\frac{\alpha^2}{2}\left(\cancel{1}+j\omega+\frac{(j\omega)^2}{2}+\cdots-\cancel{1}\right)^2 + \cdots$$

$$= 1+\alpha\left(j\omega+\frac{(j\omega)^2}{2}+\cdots\right)+\frac{\alpha^2}{2}\left((j\omega)^2+\cdots\right)+\cdots$$

Combine

COMPARE

$$= 1+\alpha j\omega + (\alpha+\alpha^2)\frac{(j\omega)^2}{2}+\cdots$$

$\Rightarrow E[X]=\alpha$, $E[X^2]=\alpha+\alpha^2$ etc.

c. $\sigma_X^2 = \alpha+\alpha^2-\alpha^2 = \underline{\underline{\alpha}} = E[X]$

4. a. Use Chebyshev $P[|X-m|\geq a] \leq \frac{\sigma^2}{a^2}$

with $a = 2\sigma$

$\Rightarrow P[|X-m|\geq 2\sigma] = \frac{\cancel{\sigma^2}}{4\cancel{\sigma^2}} = \underline{\underline{\frac{1}{4}}}$

b. For simplicity let $f_X(x) = \frac{1}{2a}$ $-a<x<a$

$\Rightarrow \mu_X = E[X] = 0$

$$\sigma_X^2 = E[X^2] = \int_{-a}^{a} x^2\frac{1}{2a}\,dx = \frac{x^3}{6a}\Big|_{-a}^{a} = \frac{a^2}{3}$$

b. $f_X(x) = 3e^{-3x} \qquad x > 0$

$$E[Y] = E[X^3] = \int_0^\infty x^3 (3e^{-3x})\,dx = 3\int_0^\infty x^3 e^{-3x}\,dx$$

NOTE: (pg. 117) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \quad n > 0$

or (pg. 573) $\int_0^\infty x^{n-1} e^{-(a+1)x}\,dx = \frac{\Gamma(n)}{(a+1)^n} \quad n > 0$

$$\Rightarrow E[Y] = 3\,\frac{\Gamma(4)}{3^4} = \frac{3!}{3^3} = \underline{\frac{2}{9}}$$

$$E[Y^2] = E[X^6] = \int_0^\infty x^6 (3e^{-3x})\,dx = 3\int_0^\infty x^6 e^{-3x}\,dx$$

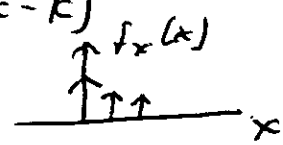
$$= 3\,\frac{\Gamma(7)}{3^7} = \frac{6!}{3^6} = \frac{80}{81} = 0.9877$$

$$\sigma_Y^2 = .9877 - \left(\tfrac{2}{9}\right)^2 = \underline{\underline{0.9383}}$$

3. Poisson pmf $\Rightarrow p_k = \alpha^k/k!\, e^{-\alpha}$

$\Rightarrow$ pdf in X $\Rightarrow f_X(x) = \sum_{k=0}^\infty \frac{\alpha^k}{k!} e^{-\alpha}\delta(x-k)$

$\Rightarrow$



a. $$\Phi_X(\omega) = \int_{-\infty}^\infty f_X(x) e^{j\omega x}\,dx = \int_{-\infty}^\infty \left[\sum_{k=0}^\infty \frac{\alpha^k}{k!} e^{-\alpha}\delta(x-k)\right] e^{j\omega x}\,dx$$

$$= \sum_{k=0}^\infty \frac{\alpha^k}{k!} e^{-\alpha} \int_0^\infty \delta(x-k) e^{j\omega x}\,dx$$

$$= \sum_{k=0}^\infty \frac{\alpha^k}{k!} e^{-\alpha} e^{j\omega k} = e^{-\alpha} \sum_{k=0}^\infty \frac{(\alpha e^{j\omega})^k}{k!}$$

NOTE: $\sum_{k=0}^\infty \frac{x^k}{k!} = e^x$

$$= e^{-\alpha} e^{\alpha e^{j\omega}} = \underline{\underline{e^{\alpha(e^{j\omega}-1)}}}$$

4b (cont). $\Rightarrow \sigma_x = \frac{a}{\sqrt{3}}$

$$P[|X - E[X]| \geq 2\sigma_x] = P[|X| \geq \tfrac{2a}{\sqrt{3}}]$$
$$= \underline{\underline{0}} \quad \text{because } \tfrac{2a}{\sqrt{3}} > a$$

c. $f_X(x) = \lambda e^{-\lambda x} u(x)$

From pg. 101: $E[X] = \frac{1}{\lambda}$, $\sigma_x^2 = \frac{1}{\lambda^2}$

$\sigma_x = \frac{1}{\lambda}$

$$P[|X - \mu_x| \geq 2\sigma] = P[|X - \tfrac{1}{\lambda}| \geq \tfrac{2}{\lambda}]$$
$$= P[X \geq \tfrac{1}{\lambda}]$$
$$= \int_{\frac{1}{\lambda}}^{\infty} \lambda e^{-\lambda x} dx$$
$$= -e^{-\lambda x}\Big|_{\frac{1}{\lambda}}^{\infty} = \underline{\underline{e^{-1}}}$$

5. a. $G_N(z) = E[z^N] = \sum_{k=0}^{\infty} p_N(k) z^k$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} z^k = \sum_{k=0}^{n} \binom{n}{k} (pz)^k q^{n-k}$$

$$= (pz + q)^n \quad \text{(binomial series)}$$
$$= [1 + p(z-1)]^n \quad (q = 1-p)$$
$$= \sum_{r=0}^{n} \binom{n}{r} p^r (z-1)^r$$

$$= \underline{\sum_{r=0}^{n} c_r \frac{(z-1)^r}{r!} \quad \text{where } c_r = \frac{n!}{(n-r)!} p^r}$$

c_r is the rth moment of the binomial RV

5b. $G_N(z) = c_0 + c_1(z-1) + c_2\frac{(z-1)^2}{2} + c_3\frac{(z-1)^3}{6} + \cdots + c_n\frac{(z-1)^n}{n!}$

$$c_r = \frac{n!}{(n-r)!}p^r$$

$$\Rightarrow \frac{d}{dz}G_N(z)\Big|_{z=1} = 0 + c_1 + 2c_2\frac{(z-1)}{2} + 3c_3\frac{(z-1)^2}{6} + \cdots \Big|_{z=1}$$

$$= c_1 = \underline{np}$$

$$\frac{d}{dz^2}G_N(z)\Big|_{z=1} = 0 + 0 + c_2 + 2c_3\frac{(z-1)}{2} + \cdots \Big|_{z=1}$$

$$= c_2 = n(n-1)p^2 = n^2p^2 - np^2$$

$$Var(x) = G_N''(1) + G_N'(1) - (G_N'(1))^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np - np^2 = \underline{\underline{np(1-p)}}$$

6. $f_x(x) = 3e^{-3x}u(x)$

a. $\Phi_x(\omega) = \int_{-\infty}^{\infty} f_x(x)e^{j\omega x}dx = \int_0^{\infty} 3e^{-3x}e^{j\omega x}dx$

$$= 3\int_0^{\infty} e^{(j\omega-3)x}dx = \frac{3}{j\omega-3}e^{(j\omega-3)x}\Big|_0^{\infty}$$

$$= \underline{\underline{\frac{3}{j\omega-3}}}$$

b. $E[x] = \frac{d\Phi_x(\omega)}{d\omega}\frac{1}{j}\Big|_{\omega=0} = \frac{+3j}{j(j\omega-3)^2}\Big|_{\omega=0} = \underline{\underline{\frac{1}{3}}}$

$E[x^2] = \frac{d^2\Phi_x(\omega)}{d\omega^2}\frac{1}{j^2}\Big|_{\omega=0} = \frac{-6}{(j\omega-3)^3}(-1)\Big|_{\omega=0} = \underline{\underline{\frac{2}{9}}}$

c. $Y = X^2 \Rightarrow x_1 = \sqrt{y}, \; x_2 = -\sqrt{y}$

$$\frac{dx_1}{dy} = \frac{1}{2\sqrt{y}}, \quad \frac{dx_2}{dy} = -\frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \sum_{i=1}^{2} f_X(x_i) \left| \frac{dx_i}{dy} \right|_{x_i = g^{-1}(y)} = 3e^{-3x} \left| \frac{1}{2\sqrt{y}} \right| + 0 \cdot \left| -\frac{1}{2\sqrt{y}} \right| \Bigg|_{x_i = g^{-1}}$$

$$= \underline{3e^{-3\sqrt{y}} \quad y \geq 0}$$

d. $Y = |X| \Rightarrow x_1 = y, \quad x_2 = -y$

$$\frac{dx_1}{dy} = 1, \quad \frac{dx_2}{dy} = -1$$

$$f_Y(y) = \sum_{i=1}^{2} f_X(x_i) \left| \frac{dx_i}{dy} \right|_{x_i = g^{-1}(y)} = 3e^{-3x} |1| + 0 \cdot |-1|$$

$$= \underline{3e^{-3y}, \; y \geq 0}$$

e. Similar to d.

$$f_Y(y) = \underline{3e^{-3y}, \; y \geq 0}$$