Transient radiation from axially symmetric sources

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A method is presented for the efficient calculation of radiated acoustic fields from a radially symmetric source in a rigid baffle excited by an arbitrary time excitation. The technique is a modal analysis based on the series expansion of the source velocity excitation in terms of either of two basis functions. Each mode is propagated by the technique with rapid convergence of the solution evident in 30 or less terms, allowing rapid and efficient computer-based solutions to be obtained. Several numerical field simulations are given.

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INTRODUCTION

Different methods now exist for computing the transient radiation or diffraction of a rigidly baffled planar source[1–8] in linear, homogeneous media. These techniques typically require the use of fast Fourier transforms (FFT's) or the evaluation of difficult integrals. A large fraction of transducers used in acoustical applications are of radial symmetry. Such symmetry allows techniques to be developed that increase the calculation efficiency of these techniques. Stepanishen,[6] Meideros and Stepanishen,[7] and Greenspan[8] have developed series expansions of the solutions for axial symmetric sources that require evaluation of integrals over limits with geometrical interpretations. The method presented in this paper is also based on the expansion of the source excitation into an infinite series over a set of orthogonal basis functions, where each of these basis functions corresponds to a vibrational mode of a circular transducer. By using the spatial frequency domain, however, a simpler expression can be found that allows rapid evaluation of the fields with a digital computer. The basic effect of propagation is to redistribute the amplitudes of the different modes. Through the development of a time-varying transfer function, it is possible to show that the redistribution is from the higher-order modes into the lower-order modes. As a consequence, the field can be simply calculated by summing the significant lower-order modes after calculation of their amplitudes (as affected by propagation).

I. BASIC THEORY

Using the result of diffraction theory, one can express the acoustic velocity potential $\phi(x,y,z,t)$ in terms of the normal velocity $v_z(x,y,0,t)$ on a planar emitting surface imbedded in a rigid baffle as[1,2]

$$\phi(x,y,z,t) = v_z(x,y,0,t) \underset{xyt}{***} [\delta(ct-R)/2\pi R], \quad (1)$$

where $*$ indicates convolution over the indicated variable and $R=(x^2+y^2+z^2)^{1/2}$. For a separable velocity given by

$$v_z(x,y,0,t) = \tau(t)s(x,y), \quad (2)$$

we have

$$\phi(x,y,z,t) = \tau(t) \underset{t}{*} \{s(x,y) \underset{xy}{**} [\delta(ct-R)/2\pi R]\}. \quad (3)$$

This latter expression is the convolution of the time excitation with the spatial impulse response,[1,2] $h(x,y,z,t)$, given as

$$h(x,y,z,t) = s(x,y) \underset{xy}{**} [\delta(ct-R)/2\pi R]. \quad (4)$$

The spatial impulse responses of a free boundary and a resilient boundary can be expressed in terms of the spatial impulse response of the rigid boundary case.[5]

To perform field calculations, it is more efficient to work in the spatial frequency domain using f_x and f_y to express the spatial frequencies. To do so, we will decompose the elementary solution [i.e., the latter part of Eq. (1)] by using the properties of the Dirac delta function. Recalling that

$$\delta[f(r)] = \sum_{i=1}^{N} \delta(r-r_i) \Big/ \left|\frac{df}{dr}\right|_{r=r_i}, \quad (5)$$

where $r=(x^2+y^2)^{1/2}$ and r_i are the N zeros of $f(r)$, we can write an expression for the outward-traveling wave as

$$\frac{\delta(ct-R)}{R} = \delta[r-(c^2t^2-z^2)^{1/2}] \Big/ \left(\frac{R(c^2t^2-z^2)^{1/2}}{ct}\right). \quad (6)$$

Taking the two-dimensional spatial transform (and recognizing that it reduces to the Hankel transform due to the radial symmetry), we have[9]

$$B[\delta(ct-R)/R] = J_0[\rho(c^2t^2-z^2)^{1/2}]H(ct-z), \quad (7)$$

where $B[\cdot]$ is the Hankel transform operator, J_0 is the zero-order Bessel function, and $\rho=(f_x^2+f_y^2)^{1/2}$.

The spatial transform $\tilde{h}(f_x,f_y,z,t)$ of the spatial impulse response of Eq. (4) can be written after substitution of Eq. (7) as

$$\tilde{h}(f_x,f_y,z,t) = \tilde{s}(f_x,f_y)J_0[\rho(c^2t^2-z^2)^{1/2}]H(ct-z), \quad (8)$$

where the multiplicative constant of $1/2\pi$ has been dropped for simplicity. (All computer-simulated fields will be normalized to the maximum value of the field.) The transfer function $J_0[\rho(c^2t^2-z^2)^{1/2}]H(ct-z)$ is now seen to be a

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time-varying spatial filter acting on the modulus of the source spectrum. This spatial filter is shown in Fig. 1. As time increases, the J_0 function decreases, thereby enhancing the lower spatial frequencies. This will cause the field to become smoother as time advances.[4,5]

For a radial distribution of source velocity $s(r)$, Eq. (8) can be written as

$$\tilde{h}(\rho,z,t) = \tilde{s}(\rho)J_0[\rho(c^2t^2 - z^2)^{1/2}]H(ct - z). \quad (9)$$

Computing the spatial impulse response from Eq. (8) or Eq. (9) normally requires performing a transform of the source velocity, multiplication by the transfer function, and an inverse transform of the product to obtain the field. For radially symmetric sources, however, the number of operations can be reduced and the field can be obtained without any Fourier or Hankel transform operation.

It is true, mathematically speaking, that a spatially bounded function can always be decomposed into a series expansion over a set of orthogonal basis functions. The wave that arrives at a plane located a distance z from the source plane will be spatially bounded. Due to causality, we know that the wave will not reach a point on the observation plane located a distance r_0 from the center of the observation plane before a time t_0, given by

$$t_0 = [z^2 + (r_0 - A)^2]^{1/2}/c, \quad (10)$$

where A is the radius of the source and c is the sound velocity. For any given time t, this equation tells us that the spatial domain of the wave in an observation plane located a distance z from the source will be between $-r_0$ and $+r_0$, where

$$r_0 = [c^2t^2 - z^2]^{1/2} + A. \quad (11)$$

For any given z and t, we need an expansion of the wave over the interval $[0,r_0]$. Since circular symmetry is present, the expansion is over Bessel functions and we can find a series that represents the spatial impulse response of the wave $h(r,z,t)$, as

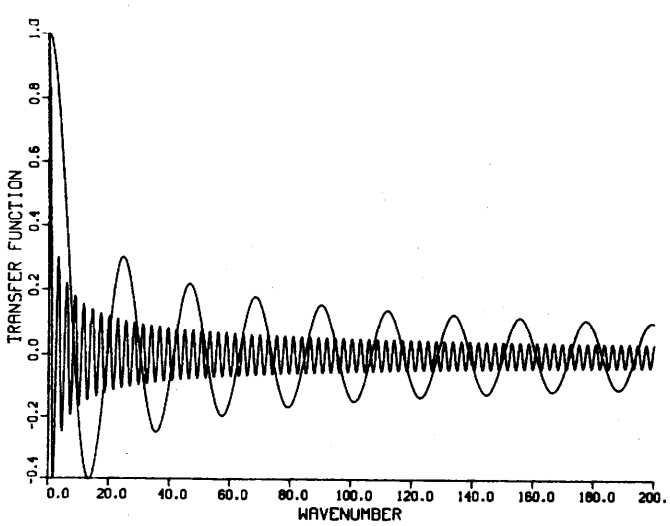

FIG. 1. Time varying transfer function for impulse temporal excitation: Large amplitude curve has $(c^2t^2 - z^2)^{1/2} = 0.0458$ and small amplitude curve has $(c^2t^2 - z^2)^{1/2} = 0.0830$.

$$h(r,z,t) = \sum_{i=1}^{\infty} h_i J_0(\alpha_i r), \quad 0 \leqslant r \leqslant r_0, \quad (12)$$

where the coefficients h_i are functions of z and t. The values of α_i depend on the particular orthogonal basis functions chosen.

Two sets of orthogonal basis functions can be profitably considered[9,10] for circular symmetry. They correspond to the roots of the following equations:

$$J_0(\alpha_i r_0) = 0 \quad (13)$$

and

$$\beta_i r_0 J_0'(\beta_i r_0) + \xi J_0(\beta_i r_0) = 0, \quad (14)$$

where J_0' is the derivative of J_0 and ξ is an arbitrary constant. The first set of eigenvalues leads to a Bessel expansion, and the second leads to a Dini expansion.[9,10] Both sets are considered below.

A. Bessel expansion

The orthogonality relation is

$$\int_0^{r_0} rJ_0(\alpha_i r)J_0(\alpha_j r)dr = \begin{cases} r_0^2 J_1^2(\alpha_i r_0)/2, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (15)$$

where α_i satisfies the relation

$$J_0(\alpha_i r_0) = 0. \quad (16)$$

For this orthogonality, the evaluation of the series expansion coefficients for the spatial impulse response is

$$h_i = 2\int_0^{r_0} rh(r)J_0(\alpha_i r)dr/r_0^2 J_1^2(\alpha_i r_0) \quad (17)$$

and the series expansion is represented by

$$h(r,0,t) = \sum_{i=1}^{\infty} h_i J_0(\alpha_i r), \quad 0 \leqslant r \leqslant r_0. \quad (18)$$

A similar expansion can be found for the radial velocity distribution at the source. Expanding again over the same integral $[0,r_0]$, we have

$$s(r) = \sum_{i=1}^{\infty} s_i J_0(\alpha_i r), \quad 0 \leqslant r \leqslant r_0, \quad (19)$$

where s_i are the weighting constants given by

$$s_i = 2\int_0^{r_0} rs(r)J_0(\alpha_i r)dr/r_0^2 J_1^2(\alpha_i r_0). \quad (20)$$

B. Dini expansion

Different sets of eigenfunctions can be generated for the eigenvalues of Eq. (14), depending on the value of ξ. For simplicity, we will assume $\xi = 0$. Hence Eq. (14) will reduce to

$$J_1(\beta_i r_0) = 0. \quad (21)$$

The orthogonality condition is

$$\int_0^{r_0} J_0(\beta_i r)J_0(\beta_j r)r\,dr = \begin{cases} r_0^2 J_0^2(\beta_i r_0)/2, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \quad (22)$$

This leads to an expansion of the spatial impulse response expressed as

$$h(r,z,t) = \sum_{i=1}^{\infty} h_i J_0(\beta_i r), \tag{23}$$

where $J_1(\beta_i r_0) = 0$ with

$$h_i = 2 \int_0^{r_0} rh(r) J_0(\beta_i r) dr / r_0^2 J_0^2(\beta_i r_0). \tag{24}$$

A similar expansion is obtained for the input velocity distribution,

$$s(r,0,t) = \sum_{i=1}^{\infty} s_i J_0(\beta_i r), \tag{25}$$

where $J_1(\beta_i r_0) = 0$ with

$$s_i = 2 \int_0^{r_0} rs(r) J_0(\beta_i r) dr / r_0^2 J_0^2(\beta_i r_0). \tag{26}$$

II. EFFICIENT FIELD CALCULATIONS

We have established the fact that the spatial source velocity term and the spatial impulse response can be expressed on the interval $[0,r_0]$ as a series of the form

$$s(r,0,t) = \sum_{i=1}^{\infty} s_i J_0(\gamma_i r), \tag{27}$$

where γ_i are the roots of either Eq. (16) or Eq. (20). We now seek a computationally efficient way to relate them. The Appendix shows that

$$h(r,z,t) = \sum_{i=1}^{\infty} s_i J_0[\gamma_i (c^2t^2 - z^2)^{1/2}] J_0(\gamma_i r) H(ct - z). \tag{28}$$

Equation (28) is the desired result that shows how each mode of the source is radiated. The

$$J_0[\gamma_i (c^2t^2 - z^2)^{1/2}] J_0(\gamma_i r)$$

term is a time-varying propagator that decreases the amplitude of the ith mode of the expansion. [The $H(ct - z)$ term ensures causality.] The filter, therefore, serves to enhance the lower-order modes. As time increases, the argument of the filtering function increases, favoring the lower-order modes.

Two special cases can be identified.[6,8] For $ct = z$, the temporal impulse response becomes

$$h(r,z,t) = \sum_{i=1}^{\infty} s_i J_0(\gamma_i r) \quad (ct = z), \tag{29}$$

which is an exact replica of the spatial distribution of the source. On axis ($r = 0$), the temporal impulse response is

$$h(0,z,t) = \sum_{i=1}^{\infty} s_i J_0[\gamma_i (c^2t^2 - z^2)^{1/2}] H(ct - z) \quad (r = 0), \tag{30}$$

which is also an image of the source spatial distribution, but it is obtained along the time axis.

III. NUMERICAL SIMULATIONS

Using Eq. (28), the field is easily computed. For each value of z, one finds the value of r_0 for each time of interest from Eq. (11). One then finds N zeros of Eq. (16) or Eq. (21). The number N required depends on the convergence of the field. Thirty values were arbitrarily chosen as adequate for these computations. (The choice was borne out by comparison with known exact solutions.) Calculation of s_i is done from either Eq. (20) or Eq. (26) and then the sum of the products in Eq. (28) is found. While the summation is theoretically infinite, the fast convergence of the series ensures that only a few terms of the series are required to evaluate the fields with sufficient accuracy. The response to a nonimpulse temporal excitation requires evaluation of the temporal convolution of Eq. (3).

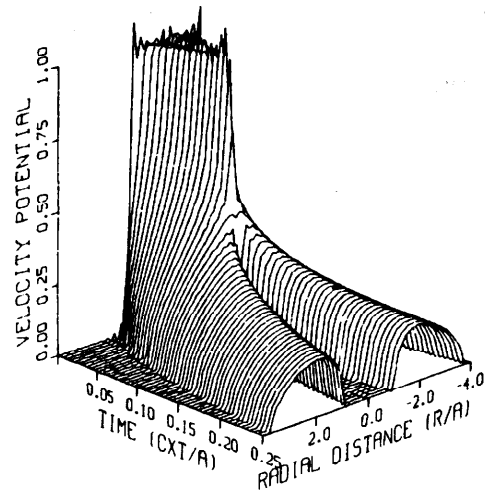

FIG. 2. Temporal impulse response for a circular piston transducer using the Bessel expansion ($A = 3.4$ cm, $z = 1$ cm).

The following plots have been evaluated using 32 points along the radial axis r and 50 points along the time axis. The plots are normalized to unit amplitude and the axes are expressed in terms of the source diameter A. All of the field patterns are observed in a plane at a distance of 10 cm from the source plane. The source radius is assumed to be 1.7 cm. The time axis begins at $t = z/c$ (i.e., when the first portion of the excited wave arrives at the observation plane). The average CPU computer time was about 6 s on an IBM 3033 mainframe computer.

Figures 2 and 3 show the field calculated for the temporal impulse response of a circular uniformly excited trans-

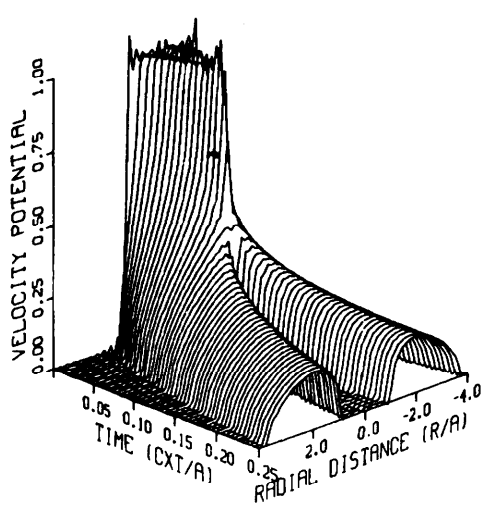

FIG. 3. Temporal impulse response for a circular piston transducer using the Dini expansion ($A = 3.4$ cm, $z = 1$ cm).

ducer using the Bessel and Dini expansions. There is no apparent difference between the representations. Despite the high spatial frequency content of the circular step wave at $t = z/c$, we note that the limited series of 30 terms still describes the field quite well. The effect of the truncation of the series is apparent at that location in the form of the "ringing" of the field in the regions of discontinuity (due to the Gibbs phenomenon). As time progresses and the high-order modes are filtered out, the convergence becomes better. The solutions of Figs. 2 and 3 compare well with closed-form solutions for the uniform piston.[1,3,6–8,10–13]

It is worth noting that there are two stages in the wave propagation through an observation plane as time increases. First, an exact replica of the shape of the source field is received (at $t = z/c$), and then is distorted as time proceeds. After a while the field splits into two portions separated in space. These are the "edge waves"[11,12] used to describe the wave propagation for the larger values of time. These waves are observed to separate further as time progresses.

Figure 4 shows the field calculated with the Dini expansion for a spatial excitation of the type

$$s(r) = J_0(7.66r/A), \tag{31}$$

corresponding to the first mode of a vibrating free circular membrane. The similarity between the source distribution and the field along the radial axis and the propagation axis is evident.

Figure 5 gives the temporal impulse response for a truncated Gaussian source excitation as calculated with the Dini expansion. The $1/e$ point is located at $r = 0.981$ cm from the center. Since the excitation consists of primarily low spatial frequencies, the effect of the propagation spatial filter on the field shape is small.

While the velocity potential is a useful quantity, it has little physical significance. One can obtain the acoustic pressure $p(x, y,z,t)$ from the potential by the relation

$$p(x, y,z,t) = \rho_0 \frac{\partial \phi}{\partial t}, \tag{32}$$

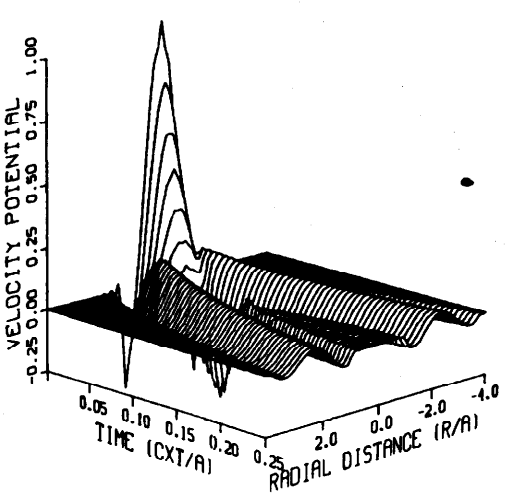

FIG. 4. Temporal impulse response for a circular transducer with a $J_0(7.66r/A)$ spatial excitation using the Bessel expansion ($A = 3.4$ cm, $z = 10$ cm).

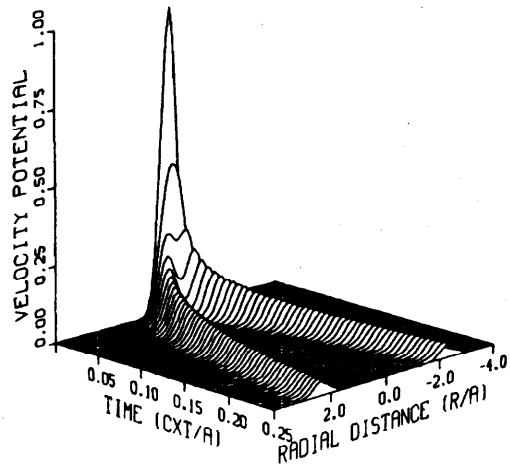

FIG. 5. Temporal impulse response for a circular transducer with a truncated Gaussian spatial excitation [$\exp - 12(r/A)^2$] using the Bessel expansion ($A = 3.4$ cm, $z = 10$ cm).

where ρ_0 is the density of the medium. Assuming axial symmetry, the acoustic pressure at an observation point will be

$$p(r,z,t) = c^2\rho_0/(c^2t^2 - z^2)^{1/2} \times \sum_{i=1}^{\infty} s_i J_1[\gamma_i(c^2t^2 - z^2)^{1/2}]J_0(\gamma_i r). \tag{33}$$

This equation provides a direct way to compute the output pressure distribution from the input velocity distribution. The computations are as easy and fast as the computation of the potential.

Figure 6 shows the pressure pattern radiated by a circular piston source as calculated using the Bessel expansion. The temporal differentiation of Eq. (32) is evident in the resultant pressure field.

IV. ARBITRARY TIME EXCITATION

For a time excitation other than an impulse, one must convolve (in the time domain) the temporal impulse response with the time-varying excitation function as in Eq.

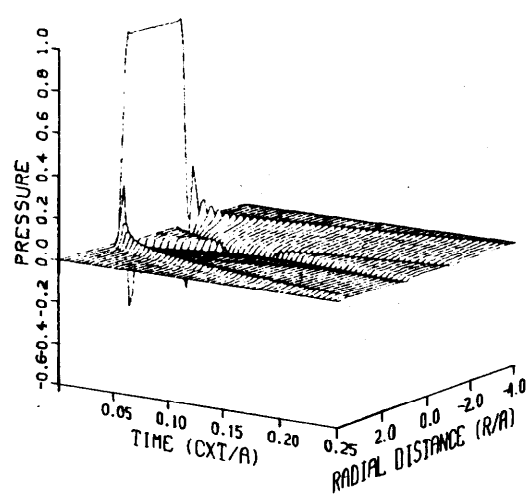

FIG. 6. Transient pressure wave from a uniform circular piston excited by a pulse (of duration $0.02A$ s) using the Bessel expansion ($A = 3.4$ cm, $z = 10$ cm).

(3). The convolution can be done either in the space–time domain or in the spatial frequency–time domain as the order of the inverse transform and the temporal convolution calculations is interchangeable.

V. SUMMARY

A method for rapidly calculating the acoustic potential field or the acoustic pressure field from an axially symmetric source has been presented. The method is based on two possible series expansions of the source velocity spatial distribution. The expansions are rapidly convergent, and therefore efficient in computation. The method does not use any Fourier or Hankel transforms; nor does it require evaluations of integrals. All operations are carried out in the space domain. The sampling intervals in time and space are independent of each other and the field solutions can be represented as desired.

ACKNOWLEDGMENTS

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APPENDIX

In this Appendix, we wish to relate the spatial impulse response $h(r,z,t)$ to the coefficients s_i of the series that represents the spatial term of the input velocity distribution. We will find this expression for the Bessel expansion. The derivation for the Dini expansion is similar. We have the Bessel expansion for $h(r,z,t)$ from Eqs. (18) and (17):

$$h(r,z,t) = \sum_{i=1}^{\infty} h_i J_0(\alpha_i r), \quad 0 \leqslant r \leqslant r_0, \tag{A1}$$

where

$$h_i = 2\int_0^{r_0} rh(r)J_0(\alpha_i r)dr/r_0^2 J_1^2(\alpha_i r_0). \tag{A2}$$

Since $h(r)$ is 0 for $r > r_0$, we can change the upper limit of the integral to ∞:

$$h_i = 2\int_0^{\infty} rh(r)J_0(\alpha_i r)dr/r_0^2 J_1^2(\alpha_i r_0). \tag{A3}$$

The Hankel (or Fourier–Bessel) transform $\tilde{f}(\rho)$ of a function $f(r)$ is defined as

$$\tilde{f}(\rho) = B[f(r)] \tag{A4}$$

$$= \int_0^{\infty} rf(r)J_0(\rho r)dr \tag{A5}$$

and hence Eq. (A3) can be written as

$$h_i = 2\tilde{h}(\alpha_i)/r_0^2 J_1^2(\alpha_i r_0). \tag{A6}$$

Since $s(r)$ is 0 for $r > A$ and since $r_0 > A$ by Eq. (11), we can show in similar fashion that the coefficients of the source expansion are given by

$$s_i = 2\tilde{s}(\alpha_i)/r_0^2 J_1^2(\alpha_i r_0). \tag{A7}$$

Finding $\tilde{s}(\alpha_i)$ from this equation, we have

$$\tilde{s}(\alpha_i) = r_0^2 J_1^2(\alpha_i r_0)s_i/2. \tag{A8}$$

From Eq. (9), we have an expression for the Hankel transform of $h(r,z,t)$ given by

$$\tilde{h}(\rho,z,t) = \tilde{s}(\rho)J_0[\rho(c^2t^2 - z^2)^{1/2}]H(ct - z). \tag{A9}$$

Substituting (A9) into (A6) and letting $\rho = \alpha_i$ as indicated, Eq. (A6) becomes

$$h_i = \frac{2\tilde{s}(\alpha_i)J_0[\alpha_i(c^2t^2 - z^2)^{1/2}]H(ct - z)}{r_0^2 J_1^2(\alpha_i r_0)}. \tag{A10}$$

Using (A8), we have

$$h_i = s_i J_0[\alpha_i(c^2t^2 - z^2)^{1/2}]H(ct - z) \tag{A11}$$

and Eq. (A1) becomes

$$h(r,z,t) = \sum_{i=1}^{\infty} s_i J_0[\alpha_i(c^2t^2 - z^2)^{1/2}] \times H(ct - z)J_0(\alpha_i r). \tag{A12}$$

This is the desired relation. The Dini expansion will produce the same result except that β_i will appear instead of α_i. Equation (A12) is repeated in the text as Eq. (28).

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