CALCULATION OF TRANSIENT RADIATION FIELDS FROM AXIAL SYMMETRIC SOURCES

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ABSTRACT

Proper understanding of the results of acoustic imaging, tissue characterization, and tomography utilizing pulsed ultrasound requires inclusion of the diffraction effects of the pulsed wave. A method is presented for the efficient calculation of pulsed ultrasonic waves from an axially symmetric source mounted in a rigid baffle and excited with an arbitrary time excitation. The technique uses a spatial modal analysis based on a series expansion of the source velocity term in either of two sets of basis functions. The choice of basis functions is arbitrary. The expansion is equivalent to a decomposition of the excitation into a set of propagation modes. Each mode is then simply propagated by the technique with rapid convergence of the solution that requires evaluation of approximately thirty (or less) terms of a series, allowing rapid computer-based solutions of the field at an object plane or at a receiver plane. Several numerical solutions are given.

INTRODUCTION

Different methods now exist to compute the transient radiation or diffraction of a rigidly baffled planar source in linear, homogeneous media (Stepanishen, 1971 and 1981; Harris, 1981a and 1981b; Guyomar and Powers, 1985 and 1986; Meideros and Stepanishen, 1984; Greenspan, 1979). These techniques typically require the use of fast Fourier transforms (FFTs) or the evaluation of difficult integrals. A large fraction of transducers used in acoustical applications possess radial symmetry. Such symmetry allows techniques to be developed that increase the the calculation efficiency of these techniques. Stepanishen (1981), Meideros and Stepanishen (1984), and Greenspan (1979) have developed series expansions of the solutions for axial symmetric sources that require evaluation of integrals over limits with geometrical interpretations. The method presented in this paper is also based on the expansion of the source excitation into an infinite series over a set of orthogonal basis functions where each of these basis functions corresponds to a vibrational mode of a circular transducer. By using the spatial frequency domain, however, a simpler expression can be found that allows rapid evaluation of the fields with a digital computer. The basic effect of propagation is to redistribute the amplitudes of the different modes. Through the development of a time-varying transfer function, it is possible to show that the redistribution is from the higher-order modes into the lower-order modes. As a consequence, the field can be simply calculated by summing the significant lower order modes after calculation of their amplitudes (as affected by propagation).

THEORY

Using the result of diffraction theory, one can express the acoustic velocity potential, $\phi(x,y,z,t)$, in terms of the normal velocity, $v_z(x,y,0,t)$, on a planar emitting surface imbedded in a rigid baffle as (Stepanishen, 1971; Harris, 1981a)

$$\phi(x,y,z,t) = v_z(x,y,0,t) \underset{xyt}{***} \frac{\delta(ct-R)}{2\pi R} \tag{1}$$

where * indicates convolution over the indicated variable and $R = \sqrt{x^2+y^2+z^2}$.

For a separable velocity given by

$$v_z(x,y,0,t) = r(t)s(x,y) \tag{2}$$

we have

$$\phi(x,y,z,t) = r(t) \underset{t}{*} \left[s(x,y) \underset{xy}{**} \frac{\delta(ct-R)}{2\pi R} \right] \tag{3}$$

This latter expression is the convolution of the time excitation with the 'spatial impulse response' (Stepanishen, 1971; Harris, 1981a), h(x,y,z,t), given as

$$h(x,y,z,t) = s(x,y) \underset{xy}{**} \frac{\delta(ct-R)}{2\pi R} \tag{4}$$

The term on the right side of the convolution is the Green's function for lossless propagation (Stepanishen, 1981) from a planar source in a rigid baffle.

To perform field calculations it is more efficient to work in the spatial frequency domain using f_x and f_y to express the spatial frequencies. To do so, we will decompose the Green's function by using the properties of the Dirac delta function. Recalling that

$$\delta(f(r)) = \sum_{i=1}^{N} \frac{\delta(r-r_i)}{|df/dr)|_{r=r_i}} \tag{5}$$

where $r = \sqrt{x^2+y^2}$ and r_i are the N zeros of f(r), we can write an expression for the outward-traveling wave (neglecting multiplicative constants) as

$$\frac{\delta(ct-R)}{R} = \frac{\delta(r-\sqrt{c^2t^2-z^2})}{|R\sqrt{c^2t^2-z^2}/ct|} \tag{6}$$

Taking the two-dimensional spatial transform (and recognizing that it reduces to the Hankel transform due to the radial symmetry), we have (Erdelyi et al., 1954)

$$B\left\{\frac{\delta(ct-R)}{R}\right\} = J_0(\rho\sqrt{c^2t^2-z^2})H(ct-z) \tag{7}$$

where $B[\cdot]$ is the Hankel transform operator, J_0 is the zero-order Bessel function, and $\rho = \sqrt{f_x^2+f_y^2}$. This transform of the Green's function is the 'propagation transfer function'.

The spatial transform, $\tilde{h}(f_x,f_y,t)$, of the spatial impulse response (Eq. 4) can be written after substitution of Eq. 7 as

$$\tilde{h}(f_x,f_y,z,t) = \tilde{s}(f_x,f_y)J_0(\rho\sqrt{c^2t^2-z^2})H(ct-z) \tag{8}$$

where the multiplicative constant of $1/2\pi$ has been dropped for simplicity. (All computer-simulated fields shown in the results will be normalized to the maximum value of the field.) The transfer function, $J_0(\rho\sqrt{c^2t^2-z^2})H(ct-z)$, is now seen to be a time-varying spatial filter acting on the modulus of the source spectrum. As time increases, the J_0 function decreases, thereby enhancing the lower spatial frequencies. This will cause the field to become smoother as time advances (Guyomar and Powers, 1985 and 1986).

For a radial distribution of source velocity, s(r), Eq. 8 can be written as

$$\tilde{h}(\rho, z, t) = \tilde{s}(\rho) J_0(\rho\sqrt{c^2t^2 - z^2}) H(ct - z) \tag{9}$$

Computing the spatial impulse response from Eq. 8 or Eq. 9 normally requires performing a transform of the source velocity, multiplication by the transfer function, and an inverse transform of the product to obtain the field. For radially symmetric sources, however, the number of operations can be reduced and the field can be obtained without any Fourier or Hankel transform evaluations.

It is true, mathematically speaking, that a spatially bounded function can always be decomposed into a series expansion over a set of orthogonal basis functions. The wave that arrives at a plane located a distance z from the source plane will be spatially bounded. Due to causality, we know that the wave will not reach a point on the observation plane located a distance r_0 from the center of the observation plane before a time t_0 given by

$$t_0 = \frac{\sqrt{z^2 + (r_0 - A)^2}}{c} \tag{10}$$

where A is the radius of the source and c is the sound velocity. For any given time t, this equation tells us that the spatial domain of the wave in the observation plane located a distance z away from the source will be between $-r_0$ and r_0 where

$$r_0 = \sqrt{c^2t^2 - z^2} + A \tag{11}$$

For any given z and t, we need an expansion of the wave over the interval $[0, r_0]$. Since circular symmetry is present, the expansion is over Bessel functions and we can find a series that represents the spatial impulse response of the wave, $h(r, z, t)$, as

$$h(r, z, t) = \sum_{i=1}^{\infty} h_i J_0(\alpha_i r) \qquad \text{for } 0 \le r \le r_0 \tag{12}$$

where the coefficients h_i are functions of z and t. The values of α_i depend on the particular orthogonal basis functions chosen.

Two sets of orthogonal basis functions can be profitably considered for circular symmetry (Erdelyi et al., 1954; Guyomar et al., 1983). They correspond to the roots of the following equations,

$$J_0(\alpha_i r_0) = 0 \tag{13}$$

and

$$\beta_i r_0 J_0'(\beta_i r_0) + \xi J_0(\beta_i r_0) = 0 \tag{14}$$

where J_0' is the derivative of J_0 and ξ is an arbitrary constant. The first set of eigenvalues leads to a Bessel expansion, and the second leads to a Dini expansion. Both sets are considered below.

1) Bessel expansion

The series expansion of the spatial impulse response is given by

$$h(r, 0, t) = \sum_{i=1}^{\infty} h_i J_0(\alpha_i r) \qquad \text{for } 0 \le r \le r_0 \tag{15}$$

where the evaluation of the series coefficients is

$$h_i = \frac{2\int_0^{r_0} r h(r) J_0(\alpha_i r)\, dr}{r_0^2 J_1^2(\alpha_i r_0)} \tag{16}$$

and α_i satisfies the relation,

$$J_0(\alpha_i r_0) = 0 \tag{17}$$

A similar expansion can be found for the radial velocity distribution at the source. Expanding again over the same interval $[0, r_0]$, we have

$$s(r) = \sum_{i=1}^{\infty} s_i J_0(\alpha_i r) \qquad \text{for } 0 \leq r \leq r_0 \tag{18}$$

where s_i are the weighting constants given by

$$s_i = \frac{2\int_0^A r s(r) J_0(\alpha_i r)\, dr}{r_0^2 J_1^2(\alpha_i r_0)} \tag{19}$$

2) Dini expansion

Different sets of eigenfunctions can be generated for the eigenvalues of Eq. 14, depending on the value of ξ. For simplicity, we will assume that $\xi = 0$. Hence, Eq. 14 will reduce to

$$J_1(\beta_i r_0) = 0 \tag{20}$$

The expansion of the spatial impulse response is

$$h(r, z, t) = \sum_{i=1}^{\infty} h_i J_0(\beta_i r) \tag{21}$$

with

$$h_i = \frac{2\int_0^{r_0} r h(r) J_0(\beta_i r)\, dr}{r_0^2 J_0^2(\beta_i r_0)} \tag{22}$$

A similar expansion is obtained for the input velocity distribution,

$$s(r, 0, t) = \sum_{i=1}^{\infty} s_i J_0(\beta_i r) \tag{23}$$

with

$$s_i = \frac{2\int_0^A r s(r) J_0(\beta_i r)\, dr}{r_0^2 J_0^2(\beta_i r_0)} \tag{24}$$

EFFICIENT FIELD CALCULATIONS

We now want to find a series expansion for the spatial impulse response in terms of the series coefficients s_i of the spatial excitation $s(r)$. The derivation is done in terms of the Bessel expansion. The Dini expansion would be similar except that β_i would appear insread of α_i. The Bessel series expansion of the spatial impulse response is given by Eq. 15,

$$h(r, 0, t) = \sum_{i=1}^{\infty} h_i J_0(\alpha_i r) \qquad \text{for } 0 \leq r \leq r_0 \tag{25}$$

with expansion coefficients given by Eq. 16,

$$h_i = \frac{2\int_0^{r_0} r h(r) J_0(\alpha_i r)\, dr}{r_0^2 J_1^2(\alpha_i r_0)} \tag{26}$$

Since h(r) is zero for $r > r_0$, we can replace the upper limit by ∞. Equation 26 becomes

$$h_i = \frac{2\int_0^{\infty} r h(r) J_0(\alpha_i r)\, dr}{r_0^2 J_1^2(\alpha_i r_0)}. \tag{27}$$

The Hankel transform $\tilde{f}(\rho)$ of a function $f(r)$ is defined by the integral,

$$\tilde{f}(\rho) = \int_0^\infty r f(r) J_0(\rho r)\, dr \tag{28}$$

and, so we can write Eq. 27 as

$$h_i = \frac{2\tilde{h}(\alpha_i)}{r_0^2 J_1^2(\alpha_i r_0)} \tag{29}$$

Similarly since $s(r)$ is zero for $r > A$ and since $r_0 \geq A$, we can write the series coefficients s_i as

$$s_i = \frac{2\tilde{s}(\alpha_i)}{r_0^2 J_1^2(\alpha_i r_0)} \tag{30}$$

We can solve this equation for $\tilde{s}(\alpha_i)$ as

$$\tilde{s}(\alpha_i) = \frac{s_i r_0^2 J_1^2(\alpha_i r_0)}{2} \tag{31}$$

From Eq. 9, we have the expression for the Hankel transform of the spatial impulse response,

$$\tilde{h}(\rho, z, t) = \tilde{s}(\rho) J_0(\rho\sqrt{c^2t^2 - z^2}) H(ct - z) \tag{32}$$

Substituting Eq. 32 into Eq. 29 and letting $\rho = \alpha_i$, as indicated, Eq. 29 becomes

$$h_i = \frac{2\tilde{s}(\alpha_i) J_0(\alpha_i\sqrt{c^2t^2 - z^2} H(ct - z)}{r_0^2 J_1^2(\alpha_i r_0)} \tag{33}$$

Substituting Eq. 31 we get

$$h_i = s_i J_0(\alpha_i\sqrt{c^2t^2 - z^2}) H(ct - z) \tag{34}$$

and Eq. 25 becomes

$$h(r, z, t) = \sum_{i=1}^{\infty} s_i J_0(\gamma_i\sqrt{c^2t^2 - z^2}) J_0(\gamma_i r) H(ct - z) \tag{35}$$

where γ_i is α_i for this Bessel expansion or β_i for the Dini expansion.

Equation 35 is the desired result that shows how each mode of the source is propagated. The $J_0(\gamma_i\sqrt{c^2t^2 - z^2}) J_0(\gamma_i r)$ term is a time-varying propagator that decreases the amplitude of the i-th mode of the expansion. (The $H(ct - z)$ term ensures causality.) The filter, therefore, serves to enhance the lower order modes. As time increases, the argument of the filtering function increases, favoring the lower order modes.

Two special cases can be identified (Greenspan, 1979; Stepanishen, 1981). For $ct = z$, the spatial impulse response becomes

$$h(r, z, t) = \sum_{i=1}^{\infty} s_i J_0(\gamma_i r) \qquad \text{for } ct = z \tag{36}$$

which is an exact replica of the spatial distribution of the source. On-axis ($r = 0$), the spatial impulse response is

$$h(0, z, t) = \sum_{i=1}^{\infty} s_i J_0(\gamma_i\sqrt{c^2t^2 - z^2}) H(ct - z) \qquad \text{when } r = 0 \tag{37}$$

which is also a replica of the source spatial distribution but it is obtained along the time axis.

NUMERICAL SIMULATIONS

Using Eq. 35, the spatial impulse response is easily computed. For each value of z, one finds the value of r_0 for each time of interest from Eq. 11. One then finds N zeros of Eqs. 17 or 20. The number N required depends on the convergence of the field. Thirty values were arbitrarily chosen as adequate for these computations. (The choice was borne out by comparison with known exact solutions.) Calculation of s_i is done from either Eq. 19 or Eq. 24 and then the sum of the products in Eq. 35 is found to produce the spatial impulse response. While the summation is theoretically infinite, the fast convergence of the series ensures that only a few terms of the series are required to evaluate the fields with sufficient accuracy. The solution to the field with a nonimpulse temporal excitation requires evaluation of the temporal convolution of Eq. 3.

The following plots have been evaluated using thirty-two points along the radial axis, r, and fifty points along the time axis. The plots are normalized to unit amplitude and the axes are expressed in terms of the source diameter, A. All of the field patterns are observed in a plane at a distance of 10 cm from the source plane. The source radius is assumed to be 1.7 cm. The time axis begins at $t = z/c$ (i.e., when the first portion of the excited waveform arrives at the observation plane). The average CPU time was about 6 seconds on an IBM 3033 mainframe computer.

To compare the fields obtained by the two expansions, Figs. 1 and 2 show the field calculated for the spatial impulse response of a circular uniformly-excited transducer using the Bessel and Dini expansions. There is no apparent difference between the representations. Despite the high spatial frequency content of the circular step wave at $t = z/c$, we note that the limited series of thirty terms still describes the field very well. The effect of the truncation of the series is apparent at that location in the form of 'ringing' of the field in the regions of discontinuity (due to the Gibbs phenomenon). As time progresses and the high-order modes are filtered out, the convergence becomes better. The fields of Figs. 1 and 2 compare well with closed-form solutions for the uniform piston (Stepanishen, 1971 and 1981; Harris, 1981b; Greenspan, 1979; Guyomar et al., 1983; Oberheltinger, 1961; Tupholme, 1969; Weight and Hayman, 1978).

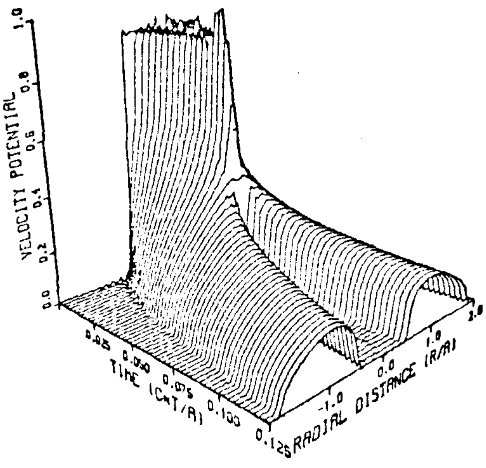

Figure 1 Spatial impulse response for a circular piston transducer using the Bessel expansion (A=3.4 cm, z=10 cm)

Figure 3, 4, and 5 illustrate the convergence of the Dini series for ten, fifteen, and twenty terms, respectively. More terms are required near $ct = 0$ due to the higher spatial frequencies there. As time progresses and the spatial frequency content is less due to the filtering action of the propagation, fewer terms are required to reach an accurate representation of the field.

Figure 6 gives the spatial impulse response for a truncated Gaussian source excitation as calculated with a Dini expansion. The $1/e$ point is located at r=0.981 cm from the center.

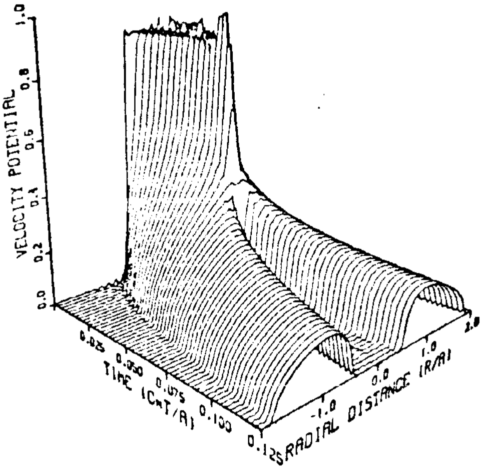

Figure 2 Spatial impulse response for a circular piston transducer using the Dini expansion (A=3.4 cm, z=10 cm)

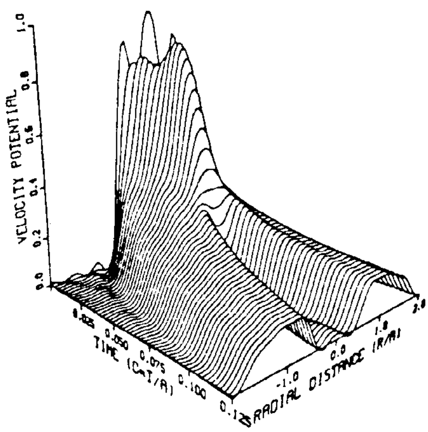

Figure 3 Ten-term Dini series impulse response for a square transducer (A=3.4 cm, z=10 cm)

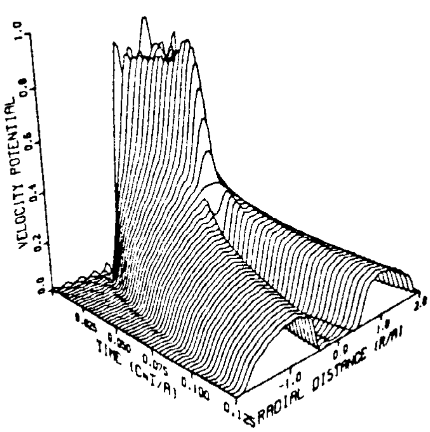

Figure 4 Fifteen-term Dini series impulse response for a square transducer (A=3.4 cm, z=10 cm)

Since the excitation consists of primarily low spatial frequencies, the effect of propagation spatial filter on the field shape is small.

While the velocity potential is a useful quantity, it has little physical significance. One can obtain the acoustic pressure, $p(x, y, z, t)$, from the potential by the relation,

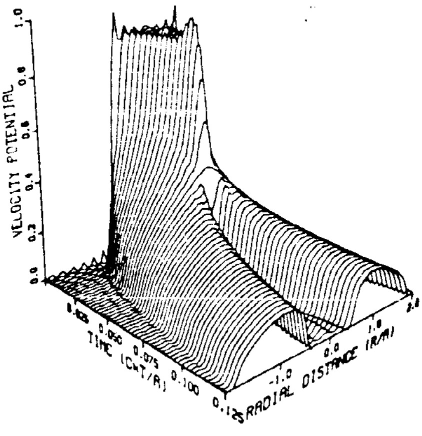

Figure 5 Twenty-term Dini series impulse response for a square transducer (A=3.4 cm, z=10 cm)

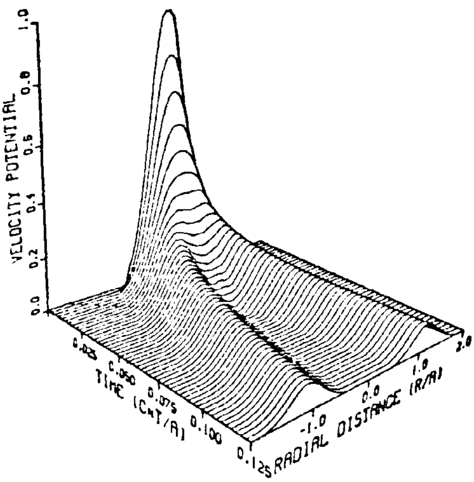

Figure 6 Spatial impulse response for a circular transducer with a truncated Gaussian spatial excitation $e^{-12(r/A)^2}$ using a Bessel expansion (A=3.4 cm, z=10 cm)

$$p(x,y,z,t) = \rho_0 \frac{\partial \phi}{\partial t} \tag{38}$$

where ρ_0 is the density of the medium. Assuming axial symmetry, the acoustic pressure at an observation point will be

$$p(r,z,t) = \frac{c^2 \rho_0}{\sqrt{c^2t^2 - z^2}} \sum_{i=1}^{\infty} s_i J_1(\gamma_i \sqrt{c^2t^2 - z^2}) J_0(\gamma_i r) \tag{39}$$

This equation provides a direct way to compute the pressure spatial impulse response from the input velocity distribution. The computations are as easy and fast as the computation of the potential.

For a time excitation other than an impulse, one must convolve (in the time domain) the spatial impulse response with the time-varying excitation function as in Eq. 3. The convolution can be done either in the space-time domain or in the spatial frequency domain as the order of the inverse spatial transform and the temporal convolution is interchangeable.

Figure 7 represents the transient pressure response of a uniform circular piston excited by a positive pulse excitation of time duration equal to 0.02A seconds. The noticeable difference between this response and the spatial impulse response for the same geometric source (Fig. 1 or 2) is due to the time derivative of Eq. 38 followed by the smoothing of the convolution of Eq. 3.

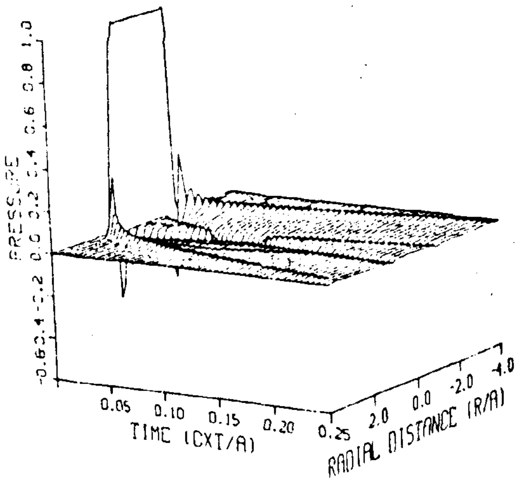

Figure 7 Transient pressure wave from a uniform circular piston excited by a pulse using the Bessel expansion (A=3.4 cm, z=10 cm)

SUMMARY

A method for rapidly calculating the acoustic potential field or the acoustic pressure field from an axially symmetric source has been presented . The method is based on two possible series expansions of the source velocity spatial distribution. The expansions are rapidly convergent, and, therefore, are efficient in calculation. The method does not use any Fourier or Hankel transforms, nor does it require evaluations of integrals (other than to obtain the s_i coefficients). All operations are carried out in the space domain. The sampling intervals in time and space are independent of each other and the field solutions can be represented as desired.

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