EC 4210 Solutions

Assignment 4

15.6. Consider the detector of the previous problem, used in a heterodyne optical receiver with $f_{IF} = 100$ MHz.

a. Calculate the local oscillator power required to make the shot noise associated with the local oscillator power equal to 15 times the shot noise associated with the dark current of the device.

b. Calculate the signal-to-noise ratio (in dB) for the receiver with P_L given by the results of part a) and $P_s = 10^{-12}$ W.

c. Calculate the minimum detectable power for the photomultiplier.

Solution: a. We want the shot noise from the local oscillator power to be 15 times the shot noise of the dark current. Hence,

$$2qG^2 \left[\frac{P_L q\eta\lambda_L}{hc}\right] B = 15 \times 2qG^2 i_d B \tag{1a}$$

and, so,

$$\frac{P_L q\eta\lambda_L}{hc} = 15 i_d \tag{1b}$$

or

$$P_L = \frac{15 i_d hc}{q\eta\lambda_L} = \frac{(15)(1 \times 10^{-12})(6.63 \times 10^{-34})(3.0 \times 10^8)}{(1.6 \times 10^{-19})(0.10)(800 \times 10^{-9})} = 2.33 \times 10^{-10}\ \mathrm{W} = 23.9\ \mathrm{nW}\,. \tag{1c}$$

Note that we used the fact that $\lambda_L \approx \lambda$.

b. Using this value of P_L and a P_s of 1×10^{-12} W, we can assume that the P_L shot noise is dominate and that we can use the results of our analysis in the notes. The S/N ratio is

$$\frac{\mathrm{S}}{\mathrm{N}} = \frac{P_s\eta\lambda}{hcB} = \frac{(1 \times 10^{-12})(0.1)(800 \times 10^{-9})}{(6.63 \times 10^{-34})(3.0 \times 10^8)} = 4.02 \times 10^5 \Rightarrow 56.0\ \mathrm{dB}\,. \tag{2}$$

c. For heterodyne detection, the minimum detectable power is

$$P_{\min} = \frac{hcB}{\eta\lambda} = \frac{(6.63 \times 10^{-34})(3.0 \times 10^8)(1)}{(0.1)(800 \times 10^{-9})} = 2.49 \times 10^{-18}\ \mathrm{W}\,. \tag{3}$$

15.7. A measurement of the spectral density of the mean-square noise current at the output of a photomultiplier tube (PMT) is 9×10^{-21} A^2·Hz^{-1} at a gain of $G = 5 \times 10^4$ when the device is unilluminated (i.e., in totally dark surroundings).

The measurement is then repeated with the PMT (at the same gain) illuminated by a constant-amplitude beam with a wavelength of 1 μm. If the measured spectral density of the mean-square noise current is 1.3×10^{-20} A^2·Hz^{-1}, calculate the power of the illuminating beam. You may assume that the PMT has a quantum efficiency of 5% at 1 μm.

Solution: We can find the dark current, since

$$S(f) = 2q(\bar{i}_c + I_d)G^2 = 2qI_dG^2 \text{ (when } \bar{i}_c = 0) \,. \tag{4a}$$

So,

$$S_1(f) = 2qI_dG^2 \tag{4b}$$

and, then,

$$I_d = \frac{S_1(f)}{2qG^2} = \frac{9 \times 10^{-21}}{2(1.6 \times 10^{-19})(5 \times 10^4)^2} = 1.125 \times 10^{-11}\text{A} \,. \tag{4c}$$

When the light is turned on, we have

$$S_2(f) = 2q(\bar{i}_c + I_d)G^2 \tag{5a}$$

and, hence,

$$\begin{aligned} \bar{i}_c &= \frac{S_2(f)}{2qG^2} - I_d = \frac{1.3 \times 10^{-20}}{2(1.6 \times 10^{-19})(5 \times 10^4)^2} - 1.125 \times 10^{-11} \\ &= 5 \times 10^{-12} \text{ A} = 5 \text{ pA} \,. \end{aligned} \tag{5b}$$

Knowing I_c, we can find the power from $I_c = P\lambda q\eta / hc$ as

$$P = \frac{hcI_c}{\lambda q \eta} = \frac{(6.63 \times 10^{-34})(3.0 \times 10^8)(5 \times 10^{-12})}{(1.6 \times 10^{-19})(1 \times 10^{-6})(0.05)} = 1.25 \times 10^{-10} \text{ W} = 125 \text{ pW} \,. \tag{6}$$

16.2. Consider a plane-wave incident on the surface of a material as shown in Figure 1 on the facing page with a power P_1. The reflection coefficient R for perpendicular incidence is $[(n_1 - n_2)/(n_1 + n_2)]^2$. The absorption coefficient is represented by α (in units of m^{-1}). Show that the power that is absorbed in a layer w deep located a distance d below the surface is

$$P(w) = P_1(1 - R) \exp(-\alpha d)[1 - \exp(-\alpha w)] \,. \tag{7}$$

Solution: At the top interface, we have reflection loss so

$$P_1 = P_i(1 - R) \,, \tag{8}$$

where R is the power reflection coefficient at the air-material boundary. The power at the top of the w-thick region is

$$P_2 = P_1 e^{-\alpha d} = P_i(1 - R)e^{-\alpha d} \tag{9a}$$

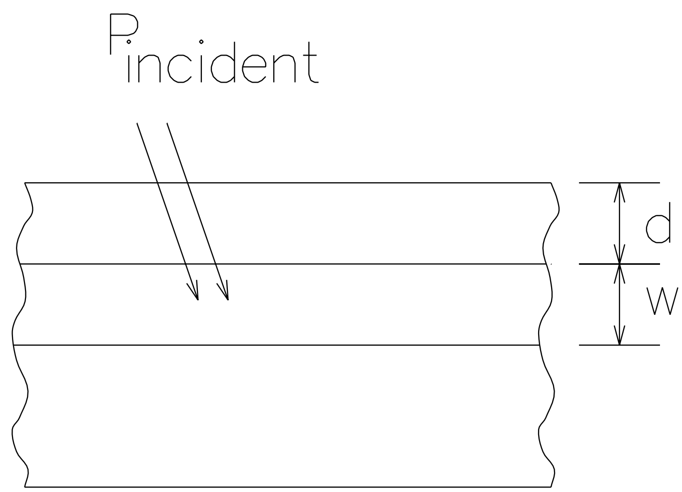

Figure 1: Geometry of Problem 2.

and that at the bottom of that region is

$$P_3 = P_2 e^{-\alpha w} = P_i(1-R)e^{-\alpha(d+w)} . \tag{9b}$$

The power absorbed in the w-thick region is the difference between the power at the top of the region and that availalable at the bottom of the region, given by $P_2 - P_3$ or

$$P(w) = P_2 - P_3 = P_i(1-R)\left[+e^{-\alpha d} - e^{-\alpha(d+w)}\right] = P_i(1-R)e^{-\alpha d}\left[1 - e^{-\alpha w}\right] . \tag{10}$$

16.3. Consider a photoconductor in the direct detection case.

a. Set up the equation for dN_c/dt.

b. Find the undetermined coefficients N_0, N_1, N_1^*, N_2, and N_2^*.

c. Find an expression for $i(t)$ in terms of k.

d. Find an expression for k.

e. Write the complete expression for $i(t)$.

f. Find an expression for $< i_s^2 >$.

g. Find an expression for $< i_N^2 >$.

h. Find an expression for S/N

i. Find an expression for the minimum detectable power.

<u>Solution:</u> a. We begin with the differential equation for the rate of change of the charge carriers

$$\frac{dN_c}{dt} = k\tilde{V}_{DD}\tilde{V^*}_{DD} - \frac{N_c}{\tau_0} \tag{11a}$$

where

$$\tilde{V}_{DD} = E_s(1 + m\cos\omega_s t)e^{j2\pi\nu t}\,. \tag{11b}$$

Substituting for $\tilde{V}_{DD}$ we have

$$\frac{dN_c}{dt} + \frac{N_c}{\tau_0} = kE_s^2\left[\left(1 + \frac{m^2}{2}\right) + 2m\cos\omega_s t + \frac{m^2}{2}\cos 2\omega_s t\right]. \tag{11c}$$

b. We assume a solution of the form:

$$N_c = N_0 + N_1 e^{j\omega_s t} + N_1^* e^{-j\omega_s t} + N_2 e^{j2\omega_s t} + N_2^* e^{-j2\omega_s t}\,. \tag{12a}$$

Substituting Eq. 12a into Eq. 11c gives

$$\begin{aligned}
& j\omega_s N_1 e^{j\omega_s t} - j\omega_s N_1^* e^{-j\omega_s t} \\
& +2j\omega_s N_2 e^{j2\omega_s t} - j2\omega_s N_2^* e^{-j2\omega_s t} \\
& +\frac{N_c}{\tau_0} + \frac{N_1 e^{j\omega_s t}}{\tau_0} + \frac{N_1^* e^{-j\omega_s t}}{\tau_0} + \frac{N_2 e^{j2\omega_s t}}{\tau_0} + \frac{N_2^* e^{-j2\omega_s t}}{\tau_0} \\
= \;& kE_s^2\left(1 + \frac{m^2}{2}\right) + \frac{2kE_s^2 m e^{j\omega_s t}}{2} + \frac{2kE_s^2 m e^{-j\omega_s t}}{2} \\
& + \frac{kE_s^2 m^2 e^{j2\omega_s t}}{4} + \frac{kE_s^2 m^2 e^{-j2\omega_s t}}{4}
\end{aligned} \tag{12b}$$

Gathering the dc terms together, we have

$$\frac{N_0}{\tau_0} = kE_s^2\left(1 + \frac{m^2}{2}\right) \tag{13a}$$

$$N_0 = kE_s^2\tau_0\left(1 + \frac{m^2}{2}\right). \tag{13b}$$

Gathering the $e^{j\omega_s t}$ terms together, we have

$$j\omega_s N_1 + \frac{N_1}{\tau_0} = kE_s^2 m \tag{14a}$$

$$N_1 = \frac{kE_s^2 m\tau_0}{1 + j\omega_s\tau_0}\,. \tag{14b}$$

Gathering the $e^{-j\omega_s t}$ terms together, we have

$$-j\omega_s N_1^* + \frac{N_1^*}{\tau_0} = kE_2^2 m \tag{14c}$$

$$N_1^* = \frac{kE_s^2 m\tau_0}{1 - j\omega_s\tau_0}\,, \tag{14d}$$

as expected.

Gathering the $e^{2j\omega_s t}$ terms together, we have

$$j2\omega_s N_2 + \frac{N_2}{\tau_0} = \frac{kE_s^2 m^2}{4} \tag{15a}$$

$$N_2 = \frac{kE_s^2 m^2 \tau_0}{4\left(1 + j2\omega_s \tau_0\right)}\,. \tag{15b}$$

Gathering the $e^{-j2\omega_s t}$ terms together, we have

$$-j2\omega_s N_2^* + \frac{N_2^*}{\tau_0} = \frac{kE_2^2 m^2}{4} \tag{16a}$$

$$N_2^* = \frac{kE_s^2 m \tau_0}{4(1 - j2\omega_s \tau_0)}\,, \tag{16b}$$

as expected.

So, the carrier number is

$$\begin{aligned} N_c =& kE_s^2 \tau_0 \Bigg[\left(1 + \frac{m^2}{2}\right) \\ &+ \frac{m\tau_0}{1 + j\omega_s \tau_0} e^{+j\omega_s t} + \frac{m\tau_0}{1 - j\omega_s \tau_0} e^{-j\omega_s t} \\ &+ \frac{m^2 \tau_0}{4(1 + j2\omega_s \tau_0)} e^{+j2\omega_s t} + \frac{m^2 \tau_0}{4(1 - j2\omega_s \tau_0)} e^{-j2\omega_s t}\Bigg]\,. \end{aligned}$$

The current is related to the N_c by

$$i(t) = \frac{qN_c(t)}{\tau_d} \tag{17}$$

$$\begin{aligned} &= kE_s^2 \frac{\tau_0}{\tau_d} \Bigg[\left(1 + \frac{m^2}{2}\right) \\ &+ \frac{m}{1 + j\omega_s \tau_0} e^{+j\omega_s t} + \frac{m}{1 - j\omega_s \tau_0} e^{-j\omega_s t} \\ &+ \frac{m^2}{4(1 + j2\omega_s \tau_0)} e^{+j2\omega_s t} + \frac{m^2}{4(1 - j2\omega_s \tau_0)} e^{-j2\omega_s t}\Bigg]\,. \end{aligned} \tag{18}$$

We can combine the $e^{j\omega_s t}$ and $e^{-j\omega_s t}$ terms and the $e^{j2\omega_s t}$ and $e^{-j2\omega_s t}$ terms in a fashion similar to the complex algebra that is used in the Appendix for the heterodyne detection case.

$$\begin{aligned} i(t) =& qkE_s^2 \frac{\tau_0}{\tau_d} \\ &\times \left[\left(1 + \frac{m^2}{2}\right) + \frac{2m}{\sqrt{1 + \omega_s^2 \tau_0^2}} \cos(\omega_s t - \phi_1) + \frac{m^2}{2\sqrt{1 + 4\omega_s^2 \tau_0^2}} \cos(2\omega_s t - \phi_2)\right]\,, \end{aligned} \tag{19}$$

where

$$\phi_1 = \tan^{-1} \omega_s \tau_0 \tag{20a}$$

$$\phi_2 = \tan^{-1} 2\omega_s \tau_0\,. \tag{20b}$$

d. We now need to find k. We know that

$$i(t)|_{m=0} = \frac{P_s q\eta\lambda}{hc}\left(\frac{\tau_0}{\tau_d}\right) = qkE_s^2\left(\frac{\tau_0}{\tau_d}\right) . \tag{21a}$$

So,

$$\begin{aligned} i(t) = &\frac{P_s q\eta\lambda}{hc}\left(\frac{\tau_0}{\tau_d}\right) \\ &\times\left[\left(1+\frac{m^2}{2}\right)+\frac{2m}{\sqrt{1+\omega_s^2\tau_0^2}}\cos(\omega_s t-\phi_1)+\frac{m^2}{2\sqrt{1+4\omega_s^2\tau_0^2}}\cos(2\omega_s t-\phi_2)\right] . \end{aligned} \tag{21b}$$

We identify the dc current as

$$\bar{i} = \frac{P_s q\eta\lambda}{hc}\left(\frac{\tau_0}{\tau_d}\right)\left(1+\frac{m^2}{2}\right) . \tag{22}$$

The ac signal current is

$$i_s(t) = \frac{P_s q\eta\lambda}{hc}\left(\frac{\tau_0}{\tau_d}\right)\frac{2m}{\sqrt{1+\omega_s^2\tau_0^2}}\cos(\omega_s t-\phi_1) . \tag{23}$$

f. The mean-square signal current, then, is

$$< i_s^2 >= \left(\frac{P_s q\eta\lambda}{hc}\right)^2\left(\frac{\tau_0}{\tau_d}\right)^2\left(\frac{2m^2}{1+\omega_s^2\tau_0^2}\right) . \tag{24}$$

g. The mean-square noise current is

$$< i_N^2 >= \frac{4q\bar{i}B\left(\frac{\tau_0}{\tau_d}\right)}{1+\omega_s^2\tau_0^2} . \tag{25}$$

h. So, the S/N ratio is

$$\frac{S}{N} = \frac{\frac{P_s^2 q^2\eta^2\lambda^2}{h^2c^2}\frac{\tau_0^2}{\tau_d^2}\frac{2m^2}{1+\omega_s^2\tau_0^2}}{\frac{4q\frac{P_s q\eta\lambda}{hc}\frac{\tau_0}{\tau_d}\left(1+\frac{m^2}{2}\right)B\frac{\tau_0}{\tau_d}}{1+\omega_s^2\tau_0^2}} \tag{26}$$

$$= \frac{P_s\eta\lambda m^2}{2hc\left(1+\frac{m^2}{2}\right)B}$$

$$= \frac{P_s\eta\lambda}{2hcB}\frac{1}{\left(\frac{1}{2}+\frac{1}{m^2}\right)} . \tag{27}$$

For $m = 1$, the signal-to-noise ratio is

$$\frac{S}{N} = \frac{P_s\eta\lambda}{2hcB}\frac{1}{\frac{3}{2}} = \frac{P_s\eta\lambda}{3hcB} . \tag{28}$$

i. We find the minimum detectable power for a photoconductor used with direct detection by letting $S/N = 1$ and finding

$$P_{\mathrm{s\ min}} = \frac{3hcB}{\eta\lambda} . \tag{29}$$