Chapter 7

Assignment 7 Solutions

5.9. An LED has a circular emitting region with a radius of 20 μm. The pattern is assumed Lambertian with an axial radiance of 80 W·cm^{-2}·sr^{-1} at a 100 ma drive current.

a. Calculate the coupling efficiency and the optical power coupled into a step–index fiber with a 140 μm core diameter and a NA $= 0.20$.

b. Repeat the calculation for a 50 μm diameter graded–index fiber with a parabolic–index profile, a center index of 1.48 and $\Delta = 1\%$.

Solution: a. In this first part we have $r_s = 20$ μm, $a = 70$ μm, and NA$= 0.20$. We note that $r_s < a$ so

$$\eta = \mathrm{NA}^2 = (0.2)^2 = 0.04 \Rightarrow 14.0 \text{ dB}. \tag{7.1}$$

The source power P_s is

$$P_s = \pi^2 B_0 r_s^2 = \pi^2 (20 \times 10^{-4})^2 (80) = 3.16 \times 10^{-3} \text{ W} = 3.16 \text{ mW}. \tag{7.2}$$

(Note that I have calculated the source area in units of cm^2; B_0 was expressed in terms of those area units.)

The power coupled into the fiber P_f is

$$P_f = \eta P_s = (0.04)(3.16{-}3) = 1.26.3 \times 10^{-4} \text{ W} = 126.3 \ \mu\text{W}. \tag{7.3}$$

b. We repeat for a graded-index fiber with $a = 25 \times 10^{-6}$, $g = 2$, $n_1 = 1.48$, and $\Delta = 1\%$. We calculate the on-axis NA(0) as

$$\mathrm{NA}(0) = n_1 \sqrt{2\Delta} = (1.48)(\sqrt{(2)(0.01)}) = 0.210. \tag{7.4}$$

We still have $r_s < a$, so

$$\eta = (\mathrm{NA}(0))^2 \left[1 - \left(\frac{2}{g+2}\right)\left(\frac{r_s}{a}\right)^g\right] \tag{7.5}$$

$$
\begin{aligned}
&= (0.21)^2 \left[1 - \left(\frac{2}{2+2}\right)\left(\frac{20}{25}\right)^2\right] \\
&= 0.030 \Rightarrow 15.23 \text{ dB}.
\end{aligned}
$$

The power coupled into the fiber P_f is

$$P_f = \eta P_s = (0.030)(3.16) = 94.8\ \mu\text{W}. \tag{7.6}$$

5.10. On the same graph plot the coupling efficiency as a function of r_s (from 0 to 50 μm) for the following step–index fibers:

a. core diameter of 50 μm and NA $= 0.15$

a. core diameter of 100 μm and NA $= 0.20$
In what regions could a lens improve the coupling efficiency?

Solution: The following equations apply.

a. We have a SI fiber with $a = 50$ μm and NA$= 0.15$. For $r_s \leq a = 25$ μm, we have

$$\eta = (\text{NA})^2 = (0.15)^2 = 0.0225, \tag{7.7}$$

and, for $r_s > a = 25$ μm, we have

$$\eta = (\text{NA})^2 \left(\frac{a}{r_s}\right)^2 = (0.0225)\left(\frac{25}{r_s}\right)^2. \tag{7.8}$$

This equation is plotted in Fig. 7.1.

b. We have a SI fiber with $a = 25$ μm and NA$= 0.20$. For $r_s \leq a = 50$ μm, we have

$$\eta = (\text{NA})^2 = (0.20)^2 = 0.040, \tag{7.9}$$

and, for $r_s > a = 50$ μm, we have

$$\eta = (\text{NA})^2 \left(\frac{a}{r_s}\right)^2 = (0.040)\left(\frac{50}{r_s}\right)^2. \tag{7.10}$$

This equation is also plotted in Fig. 7.1.

A lens would be useful anywhere on the flat portions of the curves.

5.11. Using a computer, plot the emission patterns (on a polar plot) of a Lambertian emitter and a laser emitter with $m = 10$.

Solution: See Figure 7.2.

5.12. a. Show that the power coupled into a step–index optical fiber from a source (with $r_s < a$) with a radiance of $B(\theta) = B_0(\cos\theta)^m$ is given by

$$P_f = \frac{2\pi A_s B_0 \left[1 - (\cos\theta_{\max})^{m+1}\right]}{m+1}. \tag{7.11}$$

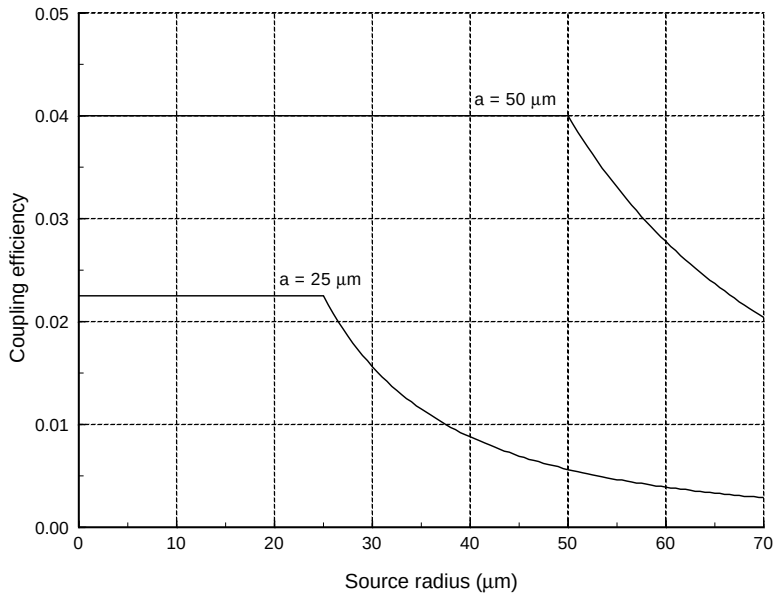

Figure 7.1: Plot for Prob. 5.10.

b. Show that the total power from the source is given by

$$P_s = \frac{2\pi A_s B_0}{m+1}\,. \tag{7.12}$$

c. Calculate the coupling efficiency and show that for small NA the answer becomes

$$\eta = \frac{(m+1)\,\mathrm{NA}^2}{2}\,. \tag{7.13}$$

Note: For a graded–index fiber the coupling efficiency is

$$\eta = \left(\frac{(m+1)\,\mathrm{NA}^2}{2}\right)\left(\frac{g}{g+2}\right)\,. \tag{7.14}$$

Solution:

a. The power in the fiber P_f is found as

$$P_f = \int\int\left[\int_0^{2\pi}\int_0^{\theta_{\max}} B_0 \cos^m\theta \sin\theta\, d\theta\, d\phi\right] d\theta_s\, r_s\, dr_s\,. \tag{7.15}$$

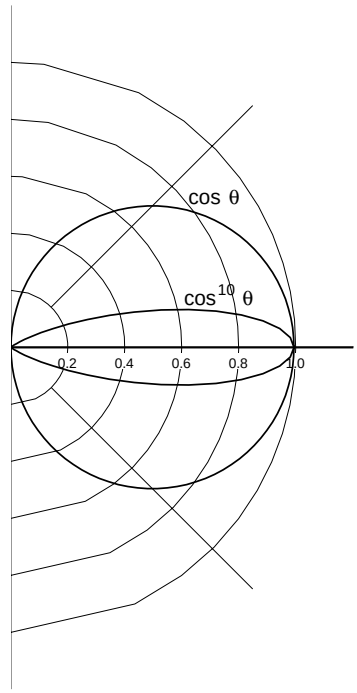

Figure 7.2: Plot for Problem 5.11.

The inner integrals are evaluated as

$$\left[\int_0^{2\pi}\int_0^{\theta_{\max}} B_0 \cos^m\theta\,\sin\theta\,d\theta\,d\phi\right] = 2\pi B_0\int_0^{\theta_{\max}}\cos^m\theta\sin\theta\,d\theta \tag{7.16}$$

$$= -2\pi B_0\,\frac{\cos^{m+1}\theta}{m+1}\bigg|_0^{\theta_{\max}} = 2\pi B_0\frac{1-\cos^{m+1}\theta_{\max}}{m+1},$$

so,

$$P_f = 2\pi B_0\left[\frac{1-\cos^{m+1}\theta_{\max}}{m+1}\right]\cdot\int\int d\theta_s\,r_s\,dr_s \tag{7.17}$$

$$= 2\pi B_0\left[\frac{1-\cos^{m+1}\theta_{\max}}{m+1}\right]\cdot A_s \qquad (\text{if } r_u = r_s)$$

$$= 2\pi B_0\left[\frac{1-\cos^{m+1}\theta_{\max}}{m+1}\right]\cdot \pi r_s^2$$

$$= \quad 2\pi^2 B_0 r_s^2 \left[\frac{1 - \cos\theta^{m+1}\theta_{\max}}{m+1} \right] .$$

b. To find P_s, we can let $\theta_{\max} \to \pi/2$ in the expression for P_f.

$$P_s = 2\pi^2 r_s^2 B_0 \left[\frac{1 - \cos^{m+1}(\pi/2)}{m+1} \right] = \frac{2\pi^2 r_s^2 B_0}{m+1} . \tag{7.18}$$

c. The coupling efficiency η is found as

$$\eta = \frac{P_f}{P_s} = 1 - \cos^{m+1}\theta_{\max} . \tag{7.19}$$

We can expand $\cos^{m+1}\theta_{\max}$ as

$$\cos^{m+1}\theta_{\max} = \left[1 - \frac{\theta_{\max}^2}{2} + \ldots \right]^{m+1} \approx \left[1 - \frac{\theta_{\max}^2}{2} \right]^{m+1} . \tag{7.20}$$

For small x, we can say that

$$\begin{aligned} (1-x)^{m+1} &= 1 - (m+1)x + \ldots \\ &\approx 1 - (m+1)x \qquad \text{(for small } x\text{)} . \end{aligned} \tag{7.21}$$

So, for small $\theta_{\max}^2/2$,

$$\cos^{m+1}\theta_{\max} \approx 1 - (m+1)\left(\frac{\theta_{\max}^2}{2} \right) \tag{7.22}$$

and

$$\eta \approx 1 - \left(1 - (m+1)\left(\frac{\theta_{\max}^2}{2} \right) \right) = \frac{(m+1)(\theta_{\max}^2)}{2} = \frac{(m+1)(\mathrm{NA}^2)}{2} , \tag{7.23}$$

where we have used $\mathrm{NA}^2 = (\sin^{-1}\theta_{\max})^2 \approx (\theta_{\max})^2$ for small values of $\theta_{\max}$.