(4) Solve differential equation

- homogeneous solution
- particular solution

need to be computed

$$\beta^2 s(t) - \frac{d^2}{dt^2} s(t) = 2\alpha\beta h(t)$$

(a) homogenous solution (assumes input $S(t) = 0$)

$$\Rightarrow \; 2\alpha\beta h_h(t) = 0 \qquad \Rightarrow \; h_h(t) = 0$$

(b) particular solution

$$h_p(t) = \frac{1}{2\alpha\beta}\left[\beta^2 s(t) - \frac{d^2}{dt^2} s(t)\right]$$

(5) Add "boundary effects" to get final solution:

– Recall:

$$\hat{h}(t) = h(t) + \underbrace{\sum_{i=0}^{2(n-m-1)} \left[ a_i \delta^{(i)}\left(t - 0^+\right) + b_i \delta^{(i)}\left(t - T + 0^+\right) \right]}_{h_b(t)}$$

– here

$$n = \text{order}\left[D\left(s^2\right)\right] \qquad D\left(s^2\right) = \sum a_k s^{2k}$$

$$m = \text{order}\left[N\left(s^2\right)\right] \qquad N\left(s^2\right) = \sum b_k s^{2k}$$

in this example:

$$N\left(s^2\right) = 2\alpha\beta \qquad m = 0$$

$$D\left(s^2\right) = -s^2 + \beta^2 \qquad n = 1$$

$$\Rightarrow \; h(t) = \sum_{i=0}^{2x(l-1)=0} a_i \delta^{(i)}\left(t - 0^+\right) + b_i \delta^{(i)}\left(t - T + 0^+\right)$$

$$= a_0 \delta^{(0)}(t) + b_0 (t - T)$$

$$h_b(t) = a_0\delta^{(0)}\left(t\right) + b_0\delta^{(0)}\left(t - T\right)$$

$$h_b(t) = a_0\delta\left(t\right) + b_0\delta\left(t - T\right)$$

need to be identified

- Computation of “boundary effect” constants $a_0$ and $b_0$
  - Replace $\hat{h}\left(t\right)$ in original integral equation

$$\hat{h}\left(t\right) = \frac{1}{2\alpha\beta}\left[\beta^2 s\left(t\right) - \frac{d^2}{dt^2}s\left(t\right)\right] + a_0\delta\left(t\right) + b_0\delta\left(t - T\right)$$

  - Original integral equation

$$\int_0^T \alpha\exp\left(-\beta\left(\left|t-\tau\right|\right)\right)h\left(\tau\right)d\tau = s\left(t\right)$$

$$\Rightarrow \int_0^T \alpha\exp\left(-\beta\left|t-\tau\right|\right)$$

$$\left(\frac{1}{2\alpha\beta}\left[\beta^2 s\left(t\right) - \frac{d^2}{dt^2}s\left(t\right)\right] + a_0\delta\left(\tau\right) + b_0\left(\tau - T\right)\right)d\tau = s\left(t\right)$$

$$= \frac{1}{2\beta}\int_0^T \exp\left(-\beta\left|t-\tau\right|\right)\left(\beta^2 s\left(t\right) - \frac{d^2}{dt^2}s\left(t\right)\right)$$

$$+\underbrace{\alpha\int_0^T \exp\left(-\beta\left|t-\tau\right|\right)\left(a_0\delta\left(t\right) + b_0\delta\left(t - T\right)\right)d\tau}_{} = s\left(t\right)$$

$\tau - 0^+$ $\qquad$ $\tau - T + 0^+$

$$\alpha\int_0^T \exp\left(-\beta\left|t-\tau\right|\right)\left(a_0\delta\left(\tau-0^+\right)+b_0\delta\left(\tau-T+0^+\right)\right)d\tau$$
$$=\alpha\left(\exp\left(-\beta t\right)\right)a_0+b_0\exp\left(+\beta\left(t-T\right)\right)$$

- Need to remove absolute value in above integral equation

$$\int_0^T \exp\left(-\beta\left|t-\tau\right|\right)d\tau$$
$$t-\tau>0 \;\Rightarrow\; t>\tau \;\Rightarrow\; \tau<t$$

$$\Rightarrow\quad \int_0^T \exp\left(-\beta\left|t-\tau\right|\right)d\tau=\int_0^t \exp\left(-\beta\left(t-\tau\right)\right)d\tau + \int_t^T \exp\left(+\beta\left(t-\tau\right)\right)d\tau$$

$\Rightarrow$ Integral equation becomes

$$\frac{1}{2\beta}\underbrace{\boxed{\int_0^t \exp\left(-\beta\left(t-\tau\right)\right)\left(\beta^2 s\left(\tau\right)-s''\left(\tau\right)\right)d\tau}}_{A}$$
$$+\frac{1}{2\beta}\underbrace{\boxed{\int_t^T \exp\left(\beta\left(t-\tau\right)\right)\left(\beta^2 s\left(\tau\right)-s''\left(\tau\right)\right)d\tau}}_{B}$$
$$+\alpha a_0\exp\left(-\beta t\right)+\alpha b_0\exp\left(\beta\left(t-T\right)\right)=s\left(t\right)$$

**Note**:

$$A = \int_0^t \exp\left(-\beta\left(t-\tau\right)\right)\left(\beta^2 s\left(\tau\right) - s''\left(\tau\right)\right) d\tau$$

$$= \exp\left(-\beta t\right) \int_0^t \exp\left(\beta\tau\right)\left(\beta^2 s\left(\tau\right) - s''\left(\tau\right)\right) d\tau$$

$$A = \beta^2 \exp\left(-\beta t\right) \int_0^t \exp\left(\beta\tau\right) s\left(\tau\right) d\tau$$

$$- \exp\left(-\beta t\right) \underbrace{\int_0^t \exp\left(\beta\tau\right) s''\left(\tau\right) d\tau}$$

- Need to integrate by parts

Recall: $\int f'g = \left[fg\right] - \int fg'$

$$f' = s''\left(\tau\right) \;\Rightarrow\; f = s'\left(\tau\right)$$

$$g = \exp\left(\beta\tau\right) \;\Rightarrow\; g' = \beta \exp\left(\beta\tau\right)$$

$$\int_0^t \exp\left(\beta\tau\right) s''\left(\tau\right) d\tau = \left[s'\left(\tau\right)\exp\left(\beta\tau\right)\right]_0^t - \int_0^t s'\left(\tau\right) \beta \exp\left(\beta\tau\right) d\tau$$

$$= s'\left(t\right)\exp\left(\beta t\right) - s'\left(0\right) - \boxed{\qquad\qquad}$$

integrate by parts

$$f' = s'\left(\tau\right) \;\Rightarrow\; f = s\left(\tau\right)$$

$$g = \exp\left(\beta\tau\right) \;\Rightarrow\; g' = \beta \exp\left(\beta\tau\right)$$

$$\int_0^t s'\left(\tau\right) \beta \exp\left(\beta\tau\right) d\tau = \beta\left[s\left(\tau\right)\exp\left(\beta\tau\right)\right]_0^t - \beta^2 \int_0^t s\left(\tau\right)\exp\left(\beta\tau\right) d\tau$$

$$= \beta s\left(t\right)\exp\left(\beta t\right) - \beta s\left(0\right) - \beta^2 \int_0^t s\left(\tau\right)\exp\left(\beta\tau\right) d\tau$$

- Replace above in expression $A$

$$A = \beta^2 \exp(-\beta t) \cancel{\int_0^t \exp(\beta\tau) s(\tau)\, d\tau}$$

$$- \exp(-\beta t)\Big[ s'(t)\exp(\beta t) - s'(0) - \beta s(t)\exp(\beta t) + \beta s(0)$$

$$+ \beta^2 \cancel{\int_0^t s(\tau)\exp(\beta\tau)\, d\tau} \Big]$$

$$\Rightarrow \quad A = -s'(t) + s'(0)\exp(-\beta t) + \beta s(t) - \beta s(0)\exp(-\beta t)$$

$$B = \int_t^T \exp\big(\beta(t-\tau)\big)\big(\beta^2 s(\tau) - s''(\tau)\big) d\tau$$

following the same type of integration by parts as for $A$ leads to:

$$B = \beta^2 \exp(\beta t) \int_t^T \exp(-\beta\tau) s(\tau)\, d\tau$$

$$- \exp(\beta t) \underbrace{\int_t^T \exp(-\beta\tau) s''(\tau)\, d\tau}_{C}$$

$$C = \int_t^T \exp(-\beta\tau) s''(\tau)\, d\tau$$

$$f' = s''(\tau) \quad \Rightarrow \quad f = s'(\tau)$$

$$g = \exp(-\beta\tau) \quad \Rightarrow \quad g' = \beta \exp(-\beta\tau)$$

$$C = \big[ s'(\tau)\exp(-\beta t) \big]_t^T - \int_t^T s'(\tau)(-\beta)\exp(-\beta\tau)\, d\tau$$

$$= s'(T)\exp(-\beta T) - s'(t)\exp(-\beta t) + \beta \underbrace{\int_t^T s'(\tau)\exp(-\beta\tau)\, d\tau}_{D}$$

**Note:**

$$D = \int_t^T s'(\tau)\exp(-\beta\tau)\, d\tau$$

$$f' = s'(t) \;\Rightarrow\; f = s(\tau)$$

$$g = \exp(-\beta\tau) \;\Rightarrow\; g' = -\beta\exp(-\beta\tau)$$

$$D = \left[s(\tau)\exp(-\beta t)\right]_t^T - \int_t^T s(\tau)(-\beta)\exp(-\beta\tau)\,d\tau$$

$$\Rightarrow\; B = \beta^2\exp(\beta t)\cancel{\int_t^T \exp(-\beta\tau)s(\tau)\,d\tau}$$

$$-\exp(\beta t)\Big[s'(T)\exp(-\beta T) - s'(t)\exp(-\beta t) + \beta s(T)\exp(-\beta T)$$

$$-\beta s(t)\exp(-\beta t) + \beta(+\beta)\cancel{\int_t^T s(\tau)\exp(-\beta\tau)\,d\tau}\Big]$$

$$\Rightarrow\; B = -\exp(\beta t)\Big[s'(T)\exp(-\beta T) - s'(t)\exp(-\beta t) + \beta s(T)\exp(-\beta T$$

$$-\beta s(t)\exp(-\beta t)\Big]$$

$$B = -s'(T)\exp\big(\beta(t-T)\big) + s'(t) - \beta s(T)\exp\big(\beta(t-T)\big) + \beta s(t)$$

$\Rightarrow$ Integral equation becomes

$$\frac{1}{2\beta}(A+B)+\alpha a_0 \exp(-\beta t)+\alpha b_0 \exp(\beta(t-T))=s(t)$$

$$\begin{aligned}A+B = & -\cancel{s'(t)}+s'(0)\exp(-\beta t)+\beta s(t)-\beta s(0)\exp(-\beta t) \\ & -s'(T)\exp(\beta(t-T))+\cancel{s'(t)}-\beta s(T)\exp(\beta(t-T))+\beta s(t) \\ = & \ 2\beta s(t)+\exp(-\beta t)\left[s'(0)-\beta s(0)\right] \\ & +\exp(\beta(t-T))\left[-s'(T)-\beta s(T)\right]\end{aligned}$$

$\Rightarrow$ Overall integral equation becomes

$$\begin{aligned}&\frac{1}{2\beta}\Big[2\beta s(t)+\exp(-\beta t)\left[s'(0)-\beta s(0)\right] \\ &\quad +\exp(\beta(t-T))\left[-s'(T)-\beta s(T)\right]\Big] \\ &\quad +\alpha a_0 \exp(-\beta t)+\alpha b_0 \exp(\beta(t-T))=s(t)\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & \cancel{s(t)}+\exp(-\beta t)\left[\frac{s'(0)}{2\beta}-\frac{s(0)}{2}+\alpha a_0\right] \\ & +\exp(\beta(t-T))\left[\alpha b_0-\frac{s'(T)}{2\beta}-\frac{s(T)}{2}\right]=\cancel{s(t)}\end{aligned}$$

$\Rightarrow$ Above expression must be valid for all *t*'s

$$\Rightarrow \begin{cases} \dfrac{s'(0)}{2\beta} - \dfrac{s(0)}{2} + \alpha a_0 = 0 \\ -\dfrac{s'(T)}{2\beta} - \dfrac{s(T)}{2} + \alpha b_0 = 0 \end{cases} \Rightarrow \boxed{\begin{aligned} a_0 &= \frac{1}{2\alpha}\left[ s(0) - \frac{s'(0)}{\beta} \right] \\ b_0 &= \frac{1}{2\alpha}\left[ s(T) + \frac{s'(T)}{\beta} \right] \end{aligned}}$$

$\Rightarrow$ Final solution is given by:

$$\boxed{\begin{aligned} \Rightarrow \quad \hat{h}(t) = \frac{1}{2\alpha\beta}\Big\{ &\big(\beta s(0) - s'(0)\big)\delta(t) \\ &+ \big(\beta s(T) + s'(T)\big)\delta(t-T) \\ &+ \beta^2 s(t) - s''(t) \Big\} \end{aligned}}$$

## ❖ Receiver Performance for Known Signals in Colored Gaussian Noise

Recall decision rule:

(1) $$G = \int_0^T y(t)\underbrace{\left(h_1(t) - h_0(t)\right)}_{h(t)} dt \;\underset{H_0}{\overset{H_1}{\gtrless}}\; T_1$$

with $$\begin{cases} H_0: & y(t) = s_0(t) + n(t) \\ H_1: & y(t) = s_1(t) + n(t) \end{cases}$$

(2) If noise is Gaussian $\Rightarrow \int_0^T y(t)h(t)\,dt$ also Gaussian

(3) Need mean and variance of $G$ only

- $$E[G]\big|_{H_0} = E\left[\int_0^T \left(s_0(t) + n(t)\right)h(t)\,dt\right]$$
  $$E_0 = \int_0^T s_0(t)h(t)\,dt$$
- $$E[G]\big|_{H_1} = \int_0^T s_1(t)h(t)\,dt = E_1$$
- $$\operatorname{var}[G]\big|_{H_0} = E\left[\left(\int_0^T \left(s_0(t) + n(t)\right)h(t)\,dt - \int_0^T s_0(t)h(t)\,dt\right)^2\right]$$
  $$= E\left[\left(\int_0^T n(t)h(t)\,dt\right)^2\right]$$

$$\Rightarrow \ \operatorname{var}\left[G\right]\Big|_{H_0} = E\left[\iint_0^T n(t)\,n(\tau)\,h(t)\,h(\tau)\,dt\,d\tau\right]$$

$$= \iint_0^T R_n(t-\tau)\,h(t)\,h(\tau)\,dt\,d\tau$$

Recall $h(t)$ can be expanded in a series (p. 19)

$$h(t) = \sum_{j=1}^{\infty} \frac{\left(s_{1j} - s_{0j}\right)}{\lambda_j} g_j(t)$$

$$\Rightarrow \ \operatorname{var}\left[G\right]\Big|_{H_0} = \iint_0^T R_n(t-\tau)$$

$$\left(\sum_{i,j}^{\infty} \frac{\left(s_{1j} - s_{0j}\right)\left(s_{1i} - s_{0i}\right)}{\lambda_j \lambda_i} g_i(t)\, g_j(t)\right) dt\, d\tau$$

$$= \sum_{i,j} \frac{1}{\lambda_i \lambda_j} \left(s_{1j} - s_{0j}\right)\left(s_{1i} - s_{0i}\right) \bullet$$

$$\int \Bigg[ \underbrace{\int_0^T R_n(t-\tau)\, g_j(\tau)\, d\tau}_{\textstyle \underbrace{\int_0^T \lambda_j\, g_j(t)\, g_i(t)\, dt}_{=0 \quad for \ i \neq j}} \Bigg] g_i(t)\, dt$$

$$\Rightarrow \operatorname{var}\left[G\right]\Big|_{H_0} = \sum_{L=1}^{\infty} \frac{1}{\lambda_i} \left(s_{1i} - s_{0i}\right)^2 = \operatorname{var}\left[G\right]\Big|_{H_1}$$

$\Rightarrow$ decision rule $\qquad G = \int_0^T y(t)h(t)\,dt \underset{H_0}{\overset{H_1}{\gtrless}} T_1$

with $\begin{cases} G \sim N\left( \int_0^T s_0(t)h(t)\,dt, \sigma_G^2 \right) \text{ under } H_0 \\ G \sim N\left( \int_0^T s_1(t)h(t)\,dt, \sigma_G^2 \right) \text{ under } H_1 \end{cases}$

with $\quad P_{FA} = Q\left( \dfrac{T_1 - E_1}{\sigma_G} \right)$

**<u>Problem</u>:** Need to compute *h*(*t*), i.e., solve integral equation.

## ❖ Whitening Filter Approach

- Whitening filter may be used to convert to white white noise case.
- Assume noise $n(t)$ has PSD $S_n(z)$

$$S_n(z) = A(z)\,A^*\left(\frac{1}{z^*}\right)$$

- Pass noise thru whitening filter $H_\omega(z)$

$$n(t) \longrightarrow \boxed{h_\omega(t)} \longrightarrow n_\omega(t)$$

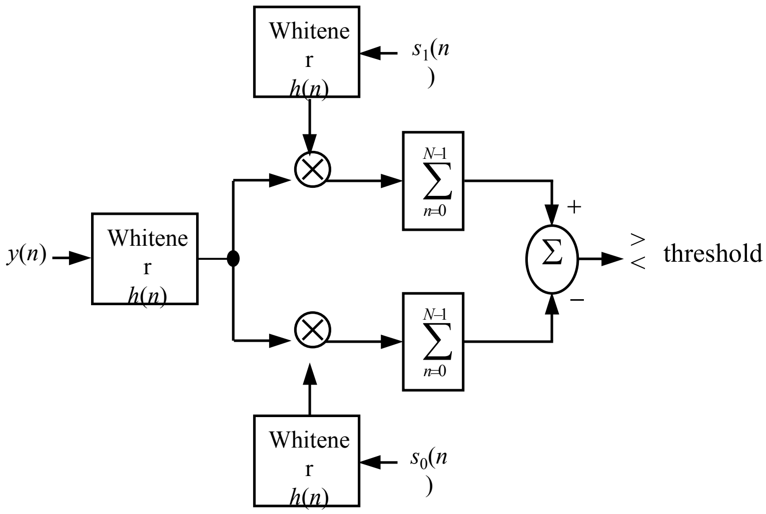
$$S_{n\omega}(z) = S_n(z)\,H_\omega(z)\,H_\omega^*\left(\frac{1}{z^*}\right)$$

if $n_\omega(t)$ is white $\quad \Rightarrow \quad S_{n\omega}(z) = \sigma_{n\omega}^2$

$$\Rightarrow \quad S_n(z)\,H_\omega(z)\,H_\omega^*\left(\frac{1}{z^*}\right) = \sigma_{n\omega}^2$$

$$\left[A(z)\,A^*\left(\frac{1}{z^*}\right)\right]H_\omega(z)\,H_\omega^*\left(\frac{1}{z^*}\right) = \sigma_{n\omega}^2$$

$$\boxed{\text{pick} \quad H_\omega(z) = A^{-1}(z)}$$


Whitener
$h(n)$
$s_1(n)$
$\sum_{n=0}^{N-1}$
+
$y(n)$
Whitener
$h(n)$
$\Sigma$
$\gtrless$
threshold
−
$\sum_{n=0}^{N-1}$
Whitener
$h(n)$
$s_0(n)$

**Example:**

$$S_n(z) = \frac{5 - 2z - 2z^{-1}}{10 - 3z - 3z^{-1}} \qquad 0.5 < |z| < 2$$

PSD to be put in the formula

$$S_n(z) = A(z)\, A^*\left(\frac{1}{z^*}\right)$$

poles and zeros inside the u.c.

poles and zeros outside the u.c.

**Example:**

Assume you are given the following colored noise data obtained as

$$s(n) = 0.6s(n-1) + v(n); \quad v(n) \sim N(0, 0.82)$$

Compute the spectral factorization of $S_s(z)$.