- Question becomes: how to compute

$$h(t) = h_1(t) - h_0(t)$$

- Recall

$$h(t) = \sum_{j=1}^{\infty} \frac{s_{1j}}{\lambda_j} g_j(t) - \frac{s_{0j}}{\lambda_j} g_j(t)$$
$$= h_1(t) - h_0(t)$$

- Note if $\lambda_j = K = 1 \ \forall j$ (white noise case)

$$h(t) = \sum_{j=1}^{\infty} s_{1j}\, g_j(t) - \sum_{j=1}^{\infty} s_{0j}\, g_j(t)$$
$$h(t) = s_1(t) - s_0(t)$$

and decision test becomes

$$\int_0^T y(t) s_1(t)\, dt - \int_0^T y(t) s_0(t)\, dt \underset{H_0}{\overset{H_1}{\gtrless}} T_1$$

- $h(t) = \sum_{j=1}^{\infty} \frac{s_{1j}}{\lambda_j} g_j(t)$ (1)

- We used an alternative expression to derive $h_1(t)$

(1) $\Rightarrow$

$$\int_0^T R_n(t-\tau) h_1(\tau) d\tau = \int_0^T R_n(t-\tau) \left[ \sum_{j=1}^{\infty} \frac{s_{1j}}{\lambda_j} g_j(t) \right] d\tau$$

$$= \sum_{j=1}^{\infty} \frac{s_{1j}}{\lambda_j} \underbrace{\int_0^T R_n(t-\tau) g_j(\tau) d\tau}_{\lambda_j g_j(t) \text{ (Mercer's theorem)}}$$

$$= \sum_{j=1}^{\infty} \frac{s_{1j}}{\cancel{\lambda_j}} \cancel{\lambda_j} g_j(t)$$

$$= \sum_{j=1}^{\infty} s_{1j} g_j(t) = s_1(t)$$

$$\Rightarrow \quad \boxed{\int_0^T R_n(t-\tau) h_1(\tau) d\tau = s_1(t)} \qquad 0 \le t \le T$$

$\Rightarrow$ $h_1(t)$ is the solution to an integral equation (called a Fredholm integral equation of the first kind).

- Integral equation solution for a rational power spectrum equation to solve is of the type:

$$\int_0^T R_n(t-\tau)\,h(\tau)\,d\tau = \delta(t) \qquad 0 \le t \le T \qquad (2)$$

Note: (1) $R_n(t)$ correlation is even for $n(t)$ real.

$\Rightarrow \; S_n(\omega) = F_T\left[R_n(t)\right]$ is even.

(2) if bounds of expression (2) were $\int_{-\infty}^{+\infty}$ we would have a convolution expression.

(3) If $S_n(\omega)$ is a valid PSD then

$$S_n(s) = L\left[R_n(\tau)\right]$$

$$= \int_{-\infty}^{+\infty} R_n(\tau)\,e^{-s\tau}\,d\tau = \frac{N(s^2)}{D(s^2)}$$

$N(s^2)$, $D(s^2)$ are polynomial of even order

$$N(s^2) = \sum_{k=0}^{n} b_k\, s^{2k}$$

$$D(s^2) = \sum_{k=0}^{m} a_k\, s^{2k}$$

assume $n < m$

(1) Assume we're dealing with:

$$\int_{-\infty}^{+\infty} R_n(t-\tau)h(\tau)d\tau = s(t)$$

$$\Rightarrow \quad R_n(t) * h(t) = s(t)$$

$$\Rightarrow \quad S_n(s) \cdot H(s) = S_S(s)$$

(2) Review: Laplace transform and end-point conditions

Property: assume you have a function defined only in [*0,T*]

$$L\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$$

Laplace transform property:

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^+) + f(T^-)\exp(-sT)$$

(3) Correspondence between differentiations and Laplace transform operation

$$s \leftrightarrow \frac{d}{dt}$$

$$s^2 \leftrightarrow \frac{d^2}{dt^2}$$

$$L[R(t)]$$

(4) $$S_s(s) = S_n(s) \cdot H(s)$$

$$S_s(s) = \frac{N(s^2)}{D(s^2)} H(s)$$

$$\Rightarrow \quad D(s^2) S_s(s) = N(s^2) H(s)$$

$$\left[ \sum_{k=0}^{m} a_k \, s^{2k} \right] S_s(s) = \left[ \sum_{k=0}^{n} b_k \, s^{2k} \right] H(s)$$

by association

$$\sum_{k=0}^{m} a_k \frac{d^{2k}}{dt^{2k}} s(t) = \sum_{k=0}^{n} b_k \frac{d^{2k}}{dt^{2k}} h(t) \qquad (3)$$

differential equation can be solved to obtain $h(t)$

(5) Need to take into account "boundary effects" due to integration defined in $[0,T]$ only.

Recall:

$$\int_0^T R_n(t-\tau) h(\tau) \, d\tau = s(t)$$

(6) Final solution can be shown to be:

*n=ord(N)*
*m=ord(D)*

$$\hat{h}(t) = h(t) + \sum_{i=0}^{2(n-m-1)} \left[ a_i \delta^{(i)}(t - 0^+) + b_i \delta^{(i)}(t - T + 0^+) \right]$$

**Example:**

Assume we have $R_n(t) = \alpha \exp\left(-\beta |t|\right)$

$$\int_0^T R_n(t-\tau) h(\tau) d\tau = s(t)$$

$$\Rightarrow \quad \int_0^T \alpha \exp\left(-\beta (t-\tau)\right) d\tau = S(t)$$

(1) Need

$$S_n(s) = L\left[R(t)\right] = L\left[\alpha \exp\left(-\beta |t|\right)\right]$$

# Review of the bilateral Laplace Transform

(1) Need $S_n(s) = L\left[R(t)\right] = \alpha\left[\alpha \exp\left(-\beta|t|\right)\right]$

$$S_n(s) = \frac{2\alpha\beta}{-s^2 + \beta^2}$$

(2) Identify $N(s^2)$ and $D(s^2)$

$$\Rightarrow \quad \begin{array}{ll} N\left(s^2\right) = 2\alpha\beta & b_0 = 2\alpha\beta; \quad b_k = 0 \quad k \neq 0 \\ D\left(s^2\right) = -s^2 + \beta^2 & a_0 = \beta^2; \quad a_1 = -1 \end{array}$$

(3) Derive differential equation

$$\sum_{k=0}^{m} a_k \frac{d^{2k}}{dt^{2k}} s(t) = \sum_{k=0}^{n} b_k \frac{d^{2k}}{dt^{2k}} h(t)$$

$$\Rightarrow \quad \beta^2 s(t) - \frac{d^2}{dt^2} s(t) = 2\alpha\beta\, h(t)$$