V Detection of Signals in Colored Gaussian Noise

- Done before: White noise
  - → not very realistic in practice
- Previous approach is difficult to extend.
- Alternative needed: use a different way to represent the signal information.
  - → series decomposition

❖ Series representation

- Several types of expansion possible
  - Fourier
  - other orthogonal polynomial functions
  - eigenfunction (KLT transform)
- What do we need?

  → an expansion in terms of a complete set of orthonormal functions with uncorrelated coefficients

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- Definitions:

  (1) A set of functions $\{g_k(t)\}_{k=1,\ldots}$ defined over the interval [0,1] is said to be orthonormal over the interval if:

$$\int_0^T g_k(t)\, g_\ell^*(t)\, dt = \delta(k-\ell)$$

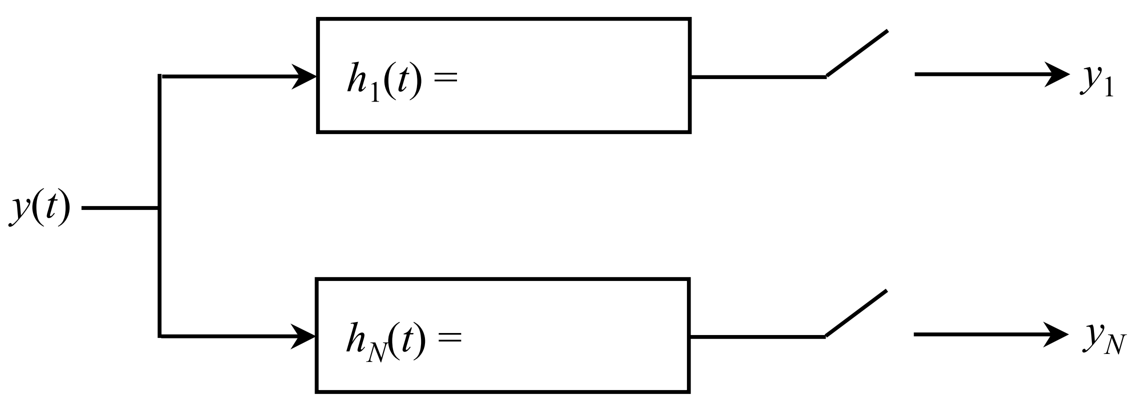
  (2) If the orthonormal set of functions $\{g_k(t)\}$ is said to be complete, then any integrable function $y(t)$ defined on the interval [0,1] may be represented as:

$$y(t) = \sum_{k=1}^{\infty} y_k g_k(t); \qquad 0 \le t \le T$$

- Properties
  - To compute y_k

$$y\left(t\right) = \qquad \sum_{k=1}^{\infty} y_k g_k\left(t\right)$$

  - Coefficients y_k may be generated as follows:



- m^{th} approximation to $y(t)$ is defined as

$$y_m(t) = \sum_{k=1}^{m} y_k g_k(t) \quad \text{such that:}$$

$$\lim_{m\to\infty} y_m(t) = y(t)$$

- Why is it useful to have y_k uncorrelated?

- How to insure the coefficients y_k are uncorrelated?
  - Assume $y(t) = s(t) + n(t)$

$n(t)$ is w.s.s. noise
zero mean.
$R_n(t,\tau) =$

  - What do we need for coefficients y_k to be said uncorrelated?

❖ Binary Detection Problem Revisited
(using expansion approach)

$$H_0: \quad y(t) = s_0(t) + n(t) \qquad 0 \le t \le T$$

$$H_1: \quad y(t) = s_1(t) + n(t) \qquad n(t) \text{ white Gausssian noise}$$

- Using projection onto a basis set $\{g_k(t)\}$

$$y_k = s_{ik} + n_k \qquad i = 0,1$$

$$y_k = \langle y(t), g_k(t) \rangle = \int y(t) g_k(t) dt$$

$$s_{1k} = \langle s_1(t), g_k(t) \rangle = \int s_1(t) g_k(t) dt$$

$$s_{0k} = \langle s_0(t), g_k(t) \rangle = \int s_0(t) g_k(t) dt$$

$$n_k = \langle n(t), g_k(t) \rangle = \int n(t) g_k(t) dt$$

- What does this mean for a 2nd order approximation?

$$y_2(t) = \sum_{k=1}^{2} s_{ik} \, g_k(t) + \sum_{k=1}^{2} n_k \, g_k(t)$$

Note:

(1) $\left\{y_k\right\}_{k=1}^{\infty}$ are RVs, why?

$$\begin{aligned} H_0: &\quad y_k = s_{0k} + n_k \\ H_1: &\quad y_k = s_{1k} + n_k \end{aligned}$$

(2) Recall $n_k = \left\langle n(t), g_k(t)\right\rangle$ is a linear transformation of $n(t)$.

$$\left.\begin{aligned} n(t) \text{ is Gaussian} &\quad \Rightarrow \quad n_k \text{ is Gaussian} \\ &\quad \Rightarrow \quad y_k \text{ is Gaussian} \end{aligned}\right\} \text{Why?}$$

- All what is needed to know about y_k is: mean & variance

- $E\left\{y_k \middle| H_1\right\} =$
- $E\left\{y_k \middle| H_0\right\} =$
- $\operatorname{var}\left\{y_k \middle| H_0\right\} = \operatorname{var}\left\{s_{0k} + n_k\right\}$

$$\begin{aligned} &= E\left\{\left(s_{0k} + n_k\right)^2\right\} - \left(E\left\{s_{0k} + n_k\right\}\right)^2 \\ &= E\left\{s_{0k}^2\right\} + E\left\{n_k^2\right\} + 2E\left\{s_{0k}.n_k\right\} \\ &\quad -\left(E\left\{s_{0k}\right\}\right)^2 \end{aligned}$$

$\Rightarrow \quad \operatorname{var}\left\{ y_k \middle| H_0 \right\} = E\left\{ n_k^2 \right\}$

- $\operatorname{var}\left\{ y_k \middle| H_1 \right\} = E\left\{ n_k^2 \right\}$ following some derivation.
- $E\left\{ n_k^2 \right\} =$

- Likelihood ratio test can be derived from orthogonal components.

$$\Lambda\left(y(t) \right) = \lim_{k \to +\infty} \Lambda\left(y_k(t) \right)$$
$$= \lim_{k \to +\infty} \frac{f_1\left(y_1 g_1(t), y_2 g_2(t), \cdots, y_k g_k(t) \right)}{f_0\left(y_1 g_1(t), y_2 g_2(t), \cdots, y_k g_k(t) \right)}$$

<u>Note</u>: Apply transformation of a RV property $Y = AX$

$$\left.\begin{array}{l} X \sim N(m_x, \sigma_x^2) \\ Y = AX \end{array}\right\} \Rightarrow \; Y \sim \left(Am_x, A^2\sigma_x^2 \right)$$

Note:

$$f_0\left(y_n\right) \sim N\left(s_{0n}, \sigma_n^2\right) \Rightarrow \; f_0\left(y_n g_n\left(t\right)\right) \sim N\left(g_n\left(t\right)s_{0n}, g_n^2\left(t\right)\sigma_n^2\right)$$

$$\Rightarrow \; f_0\left(y_n K\right) \sim N\left(Ks_{0n}, K^2\sigma_n^2\right)$$

Thus:

$$\frac{f_1\left(y_n g_n\left(t\right)\right)}{f_0\left(y_n g_n\left(t\right)\right)} = \frac{\dfrac{1}{K\sigma_n\sqrt{2\pi}} \exp\left(-\dfrac{1}{2\sigma_n^2 K^2}\left(Ky_n - Ks_{1n}\right)^2 \right)}{\dfrac{1}{K\sigma_n\sqrt{2\pi}} \exp\left(-\dfrac{1}{2\sigma_n^2 K^2}\left(Ky_n - Ks_{0n}\right)^2 \right)}$$

$$= \frac{\dfrac{1}{\sigma_n\sqrt{2\pi}} \exp\left(-\dfrac{1}{2\sigma_n^2}\left(y_n - s_{1n}\right)^2 \right)}{\dfrac{1}{\sigma_n\sqrt{2\pi}} \exp\left(-\dfrac{1}{2\sigma_n^2}\left(y_n - s_{0n}\right)^2 \right)}$$

$$= \frac{f_1\left(y_n\right)}{f_0\left(y_n\right)}$$

$$\Rightarrow \quad \Lambda\left(y(t)\right) = \lim_{k \to +\infty} \frac{f_1\left(y_s, \cdots, y_k\right)}{f_0\left(y_s, \cdots, y_k\right)}$$

$$= \lim_{k \to +\infty} \prod_{n=1}^{k} \frac{f_1\left(y_n\right)}{f_0\left(y_n\right)}$$

$$= \lim_{k \to +\infty} \prod_{n=1}^{k} \frac{\exp\left(-\frac{1}{2\sigma_n^2}\left(y_n - s_{1n}\right)^2\right)}{\exp\left(-\frac{1}{2\sigma_n^2}\left(y_n - s_{0n}\right)^2\right)}$$

$$\Rightarrow \quad \Lambda\left(y(t)\right) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_0$$

$$Ln\left(\Lambda\left(y(t)\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

$$Ln\Big(\Lambda\big(y(t)\big)\Big) \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

$$\Rightarrow -\frac{1}{2\sigma_n^2}\left(\sum_{n=1}^{k}\underbrace{\left(y_n - s_{1n}\right)^2 - \left(y_n - s_{0n}\right)^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

$$\left(\cancel{y_n} - s_{1n} - \cancel{y_n} + s_{0n}\right)\left(y_n - s_{1n} + y_n - s_{0n}\right)$$

$$= \left(s_{0n} - s_{1n}\right)\left(2y_n - s_{0n} - s_{1n}\right)$$

$$= -\left(s_{0n} - s_{1n}\right)\left(s_{0n} + s_{1n}\right) + 2y_n\left(s_{0n} - s_{1n}\right)$$

$$\Rightarrow -\frac{1}{2\sigma_n^2}\sum_{n=1}^{K}\left(-\left(s_{0n}^2 - s_{1n}^2\right) + 2y_n\left(s_{0n} - s_{1n}\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

$$\frac{1}{\sigma_n^2}\sum_{n=1}^{k}\left(\frac{s_{0n}^2 - s_{1n}^2}{2} + y_n\left(s_{1n} - s_{0n}\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

- Recall $\sigma_n^2 = \frac{N_0}{2}$

$$\Rightarrow \frac{1}{N_0}\sum_{n=1}^{k}\left(s_{0n}^2 - s_{1n}^2\right) + \frac{2}{N_0}\sum_{n=1}^{k} y_n\left(s_{1n} - s_{0n}\right) \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

- As $k \to +\infty \qquad \sum_{n=1}^{k} \to \int_0^T$

$$Ln\Big(\Lambda\big(y(t)\big)\Big) = \lim_{k\to+\infty} Ln\big(y_k(t)\big)$$

$$= \frac{1}{N_0}\int_0^T \left(s_0^2(t) - s_1^2(t)\right)dt + \frac{2}{N_0}\int_0^T y(t)\left(s_1(t) - s_0(t)\right)dt \gtrless Ln(\lambda_0)$$

$$\Rightarrow \boxed{\int_0^T y(t)s_1(t)\,dt - \int_0^T y(t)s_0(t)\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}Ln(\lambda_0) + \frac{1}{2}\int_0^T\left(s_1^2(t) - s_0^2(t)\right)dt}$$

❖ Karhunen-Loève Expansion

- Definition: The KL expansion is based on the eigenfunctions of the noise autocorrelation function.
- Properties
  - The eigenfunctions $\{g_k(t)\}$ form a complete orthonormal set

$$\int_0^T g_n(t)\, g_m(t)\, dt = \delta(n-m)$$

  - Noise and received signal coefficients are uncorrelated

    Recall: $\quad n_k = \int_0^T n(t)\, g_k(t)\, dt$

$$E\{n_k n_m\} = E\left[\left(\int_0^T n(t)\, g_k(t)\, dt\right)\left(\int_0^T n(t)\, g_m(t)\, dt\right)\right]$$

$$= E\left[\int_0^T \int n(t)\, n(\tau)\, g_k(t)\, g_m(\tau)\, dt\, d\tau\right]$$

$$= \int_0^T \int \underbrace{E\left[n(t)\, n(\tau)\right]}_{R_n(t-\tau)} g_k(t)\, g_m(\tau)\, dt\, d\tau$$

$$= \int_0^T \left[\int_0^T R_n(t-\tau)\, g_m(\tau)\, d\tau\right] g_k(t)\, dt$$

Suppose we have:

$$\int_0^T g_m(\tau) R_n(t-\tau)\, d\tau = \lambda_m g_m(t) \qquad (1)$$

In such a case:

$$\int_0^T \int R_n(t-\tau) g_m(\tau) g_k(t)\, dt\, d\tau = \int_0^T \lambda_m g_m(t) g_k(t)\, dt$$

$$= \begin{cases} \lambda_m & m \neq k \\ 0 & m = k \end{cases}$$

Therefore:

if (1) holds, then (n_k, n_m) and (y_k, y_m) for $k \neq m$

are uncorrelated and their variance is λ_m

❖ Mercer Theorem

Any real valued function $R(t,s)$ symmetric, PSD, may be expanded into a series:

$$R(t,s) = \sum_{k=1}^{\infty} \lambda_k g_k(t) g_k(s) \qquad 0 \le t,s \le T$$

with

$$\lambda_k g_k(t) = \int_0^T R(t,s) g_k(s)\, ds$$

where $\{g_k(t)\}$ is a complete orthonormal basis set.

Applying Mercer s theorem' to

$$E\{n_k n_m\} = \int_0^T \left[\underbrace{\int_0^T R_n(t_2\tau) g_m(\tau)\, d\tau}_{\lambda_m g_m(t)} \right] g_k(t)\, dt$$

$$= \int_0^T \lambda_m g_m(t) g_k(dt)\, dt = \lambda_m \delta(m-k)$$

$$\Rightarrow \quad E\{n_k n_m\} = \lambda_m \delta(m-k)$$

$\Rightarrow$ Noise/received sample coefficients are uncorrelated when using the KL expansion.

Gaussian noise $\Rightarrow$ noise samples are independent.

❖ Detection of Known Signals in Additive Colored Gaussian Noise

$$y(t) = s_i(t) + n(t) \qquad i = 0, 1, \cdots$$

$n(t)$: Gaussian w.s.s. noise

- Take as sampled of received signal the coefficients y_k of the KL expansion.

$$y_k = \int_0^T y(t)\, g_k(t)\, dt$$
$$n_k = \int_0^T n(t)\, g_k(t)\, dt \qquad k = 1, 2, \cdots$$

under H_0: $\quad y_k = s_{0k} + n_k$

under H_1: $\quad y_k = s_{1k} + n_k \qquad k = 1, 2, \ldots$

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$$
\begin{aligned}
H_0: &\quad y_j = s_{0j} + n_j \\
H_1: &\quad y_j = s_{1j} + n_j
\end{aligned}
\qquad\qquad j = 1, 2, \cdots
$$

$$E\left\{ y_j \middle| H_0 \right\} =$$

$$E\left\{ y_j \middle| H_1 \right\} =$$

$$\operatorname{var}\left\{ y_j \middle| H_0 \right\} =$$

$$\operatorname{var}\left\{ y_j \middle| H_1 \right\} =$$

$$\Lambda\left(y_k\left(t\right)\right) = \frac{f_1\left\{ y_1, \cdots, y_k \right\}}{f_0\left\{ y_1, \cdots, y_k \right\}}$$

$$= \frac{\prod_{j=1}^{k} \left(\frac{1}{2\pi\lambda_j} \right)^{1/2} \exp\left(\frac{-(y_j - s_{1j})^2}{2\lambda_j} \right)}{\prod_{j=1}^{k} \left(\frac{1}{2\pi\lambda_j} \right)^{1/2} \exp\left(\frac{-(y_j - s_{0j})^2}{2\lambda_j} \right)}$$

Note:

$$\frac{-1}{2\lambda_{\mathrm{j}}}\left((\mathrm{y}_{\mathrm{j}}-s_{1j})^2-(\mathrm{y}_{\mathrm{j}}-s_{0j})^2\right)=$$

$$\underline{\text{Note}}: \quad y_j = \int_0^T y(t) g_j(t)\,dt \qquad s_j = \int_0^T s(t) g_j(t)\,dt$$

$\Rightarrow$

$$Ln\Big(\Lambda\big(y_k(t)\big)\Big) = \sum_{j=1}^{k} \frac{y_j}{\lambda_j}\left(s_{1j} - s_{0j}\right) + \frac{1}{2}\sum_{j=1}^{k} \frac{\left(s_{0j}^2 - s_{1j}^2\right)}{\lambda_j} \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

$$= \sum_{j=1}^{k} \frac{1}{\lambda_j}\left(\int_0^T y(t_1) g_j(t_1)\,dt_1\right)\left(\int_0^T (s_1(t_2) - s_0(t_2)) g_j(t_2)\,dt_2\right)$$

$$+ \frac{1}{2}\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big[\int_0^T \int s_0(t') s_0(\tau) g_j(t') g_j(\tau)\,dt'd\tau$$

$$- \int_0^T \int s_1(t') s_1(\tau) g_j(t') g_j(\tau)\,dt'd\tau\Big] \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

As k $\rightarrow \infty$

$$Ln\big(\Lambda(y(t)\big) =$$

$$\int_0^T \int y(t_1)\big(s_1(t_2) - s_0(t_2)\big) \sum_{j=1}^{\infty} \frac{1}{\lambda_j} g_j(t_1) g_j(t_2)\,dt_1 dt_2$$

$$\boxed{+ \frac{1}{2}\int_0^T \int \big(s_0(t') s_0(\tau) - s_1(t') s_1(\tau)\big) \sum_{j=1}^{\infty} \frac{1}{\lambda_j} g_j(t') g_j(\tau)\,dt'd\tau} \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)$$

independent of received data = K

$$\int_0^T \int y(t_1)(s_1(t_2)-s_0(t_2))\sum_{j=1}^{\infty}\frac{1}{\lambda_j} g_j(t_1) g_j(t_2)\, dt_1 dt_2 \underset{H_0}{\overset{H_1}{\gtrless}} Ln(\lambda_0)\text{-K} = T_1$$

$$\int_0^T y(t_1)\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\int_0^T \left(s_1(t_2)-s_0(t_2)\right) g_j(t_2) g_j(t_1)\, dt_1 dt_2 \underset{H_0}{\overset{H_1}{\gtrless}} T_1$$

$$\int_0^T y(t_1)\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\int_0^T \left(s_1(t_2)-s_0(t_2)\right) g_j(t_2)\, dt_2\, g_j(t_1)\, dt_1 \underset{H_0}{\overset{H_1}{\gtrless}} T_1$$

$$\Rightarrow \int_0^T y(t_1)\underbrace{\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\left(s_{1j}-s_{0j}\right) g_j(t_1)}_{h(t_1)}\, dt_1 \underset{H_0}{\overset{H_1}{\gtrless}} T_1$$

$$\boxed{\begin{aligned}
&\Rightarrow \int_0^T y(t_1)\left(\underbrace{\sum_{j=1}^{\infty}\frac{s_{1j}}{\lambda_j} g_j(t_1)}_{h_1(t_1)} - \underbrace{\sum_{j=1}^{\infty}\frac{s_{0j}}{\lambda_j} g_j(t_1)}_{h_0(t_1)}\right) dt_1 \underset{H_0}{\overset{H_1}{\gtrless}} T_1 \\
&\Rightarrow \int_0^T y(t_1)\left(h_1(t_1)-h_0(t_1)\right) dt_1 \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad T_1
\end{aligned}}$$

$\Rightarrow$ Note: correlator structure

$$\int_0^T y(t_1)\big(h_1(t_1) - h_0(t_1)\big)\,dt_1 \underset{H_0}{\overset{H_1}{\gtrless}} \mathrm{T}_1$$