VI. Estimation

- Until now: Concerned about signal present or not.
- Goal: Identify parameters of interest.
- Estimates considered:
  - MAP
  - ML
  - Bayes

(1) Maximum A Posteriori (MAP) estimate

Pick for parameter estimate that leads to maximum a posteriori density.

$$\alpha_{\text{estimate}} = \max_{\{\alpha\}} f\left(\alpha \middle| y\right)$$

$$= \max_{\{\alpha\}} \left[\frac{f\left(y \middle| \alpha\right) f\left(\alpha\right)}{f\left(y\right)} \right]$$

❖ Example:

- Assume you have

$$y_i = a + n_i \qquad i = 0, \cdots, N-1$$

  - $n_i \sim N\left(0, \sigma_n^2\right)$ i.i.d.
  - $a \sim N\left(0, \sigma_a^2\right)$

- Find MAP estimate of a

(2) Maximum Likelihood (ML) estimation

Pick for parameter estimate that which maximizes the likelihood function $f(y|\alpha)$

$$\alpha_{ML} = \max_{\{\alpha\}} f\left(y \middle| \alpha\right)$$

Example: given $y_i = \alpha + n_i \qquad i = 0, \ldots, N-1$

$$n_i \sim N\left(0, \sigma_n^2\right) \qquad \text{i.i.d.}$$

find a_{ML}

(3) Bayes estimate

- Take into account the cost of making an estimation even.

  Assume estimated value of α is $\hat{\alpha}$

  Possible cost function case:

$$
\begin{aligned}
C(\alpha,\hat{\alpha}) &= |\alpha-\hat{\alpha}| \\
&= (\alpha-\hat{\alpha})^2 \\
&= (\alpha-\hat{\alpha}) \qquad \text{etc.}\cdots
\end{aligned}
$$

- Goal: find the estimated parameter which minimizes overall cost.
- Overall cost is defined as:

$$C(\hat{\alpha}) = \int C(\hat{\alpha}|\underset{\substack{\uparrow \\ \text{observation}}}{y})\, f(y)\, dy$$

where:

$$C(\hat{\alpha}\mid y) = \int C\left(\alpha,\widehat{\alpha}\right) f(\alpha\mid y)\, d\alpha \quad \leftarrow$$ cost of estimate α given observation y

Example:

- Assume $C(\hat{\alpha},\alpha)=(\hat{\alpha}-\alpha)^2$

$$C(\hat{\alpha}|y)=\int(\hat{\alpha}-\alpha)^2 f(\alpha|y)\,d\alpha$$

- $C(\hat{\alpha})=\int C(\hat{\alpha}|y)f(y)\,dy$ is minimized if

$C(\hat{\alpha}|y)$ is minimized

$\Downarrow$

$C(\hat{\alpha}|y)$ is minimized for $\frac{\partial}{\partial\alpha}C(\hat{\alpha}|y)=0$

$$\Downarrow\ \frac{\partial}{\partial\hat{\alpha}}C(\hat{\alpha}|y)=\frac{\partial}{\partial\hat{\alpha}}\int(\hat{\alpha}-\alpha)^2 f(\alpha|y)\,d\alpha$$

Example

Assume the signal x has pdf $f(x)=\exp(-x)u(x)$
Assume the noise n has pdf $f(n)=2\exp(-2n)u(n)$
Assume the received signal $y=x+n$

Compute the Bayes estimate of x given y is received

❖ Properties of Estimators

- estimates are functions of the data
- data has random components
- estimates are RVs

$\Rightarrow$ estimates have statistical properties.

- Definition 1: $\hat{\alpha}$ is an unbiased estimate of α if

$$E\{\hat{\alpha}\} = \alpha$$

estimate true parameter

- Definition 2: $\hat{\alpha}$ is an asymptotically unbiased estimate if:

$$\lim_{N \to +\infty} E\{\hat{\alpha}_N | \alpha\} = \alpha$$

- Definition 3: the parameter estimate $\hat{\alpha}$ of α is said to be a consistent estimate if:

$$\lim_{N \to +\infty} \Pr\{(\hat{\alpha} - \alpha) > \varepsilon\} = 0$$

- Definition 4: the unbiased estimate $\hat{\alpha}$ is called the minimum variance estimate $\hat{\alpha}_{MV}$ if:

$$\mathrm{var}(\hat{\alpha}_{MV}) \le \mathrm{var}(\hat{\alpha}) \qquad \text{any other } \hat{\alpha}$$

- Definition 5: the Cramer-Rao bound bounds the variance of the estimate $\hat{\alpha}$ as:

$$\sigma_{\hat{\alpha}}^2 \gtrless \frac{1}{E\left\{\left(\frac{\partial \alpha_n f\left(y|\alpha\right)}{\partial \alpha}\right)^2\right\}} = \frac{1}{E\left\{\frac{\partial^2}{\partial \alpha^2}\alpha_n f\left(y|\alpha\right)\right\}}$$

- Definition 6: an estimate $\hat{\alpha}$ of the parameter α is called an efficient estimate when it meets the CR bound.

❖ Application to Estimation of Signal Parameters in Additive White Noise

- $y(t) = s(t,\alpha) + n(t); \qquad 0 \le t \le T$

  unknown parameters to be estimated (amplitude, phase, frequency, etc.)

- ML estimate given by:

$$\hat{\alpha}_{ML} = \max_{\{\alpha\}} f\left(y \middle| \alpha\right)$$

$$= \max_{\{\alpha\}} \left(K \exp\left(-\frac{1}{N_0}\right) \int_0^T \frac{\left(y(t) - s(t,\alpha)\right)^2}{2\sigma^2} dt \right)$$

maximization can be replaced with minimization

$$= \min_{\{\alpha\}} \left(\int_0^T \left(y(t) - s(t,\alpha)\right)^2 dt \right)$$

$$\Rightarrow \qquad \frac{\partial}{\partial \alpha} \left(\int_0^T \left(y(t) - s(t,\alpha)\right)^2 dt \right)$$

$$= \int_0^T \frac{\partial}{\partial \alpha} \left(y(t) - s(t,\alpha)\right)^2 dt$$

$$= \int_0^T -\left(y(t) - s(t,\alpha)\right) \frac{\partial}{\partial \alpha} s(t,\alpha)\, dt$$

$\Rightarrow$ $\hat{\alpha}$ obtained as α which satisfies

$$\int_0^T -\left(y(t) - s(t,\alpha)\right)\frac{\partial}{\partial\alpha}\left(s(t,\alpha)\right)dt = 0$$

(a) Assume $s(t,\alpha) = \alpha s(t)$

$$= A\cos\left(\omega_0 t + \alpha\right)$$

$$= s(t-\alpha)$$