IV.b Detection of Dynamic Signals in White Gaussian Noise

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❖ Detection of Signals with Random Parameters

- Fact: In practical applications not all signal characteristics may be known.
  → phase, amplitude, frequency may be random
- Assume *a priori* pdfs are known.

(1) <u>Signals with random phase</u>

– Problem to solve

$$\begin{aligned} H_0 &: \quad s_0(t) = 0 \\ H_1 &: \quad s_1(t) = A\sin(\omega_c t + \theta) \end{aligned} \qquad 0 \le t \le T$$

– Problem to solve at the receiver

$$\begin{aligned} H_0 &: \quad y(t) = n(t) \\ H_1 &: \quad y(t) = A\sin(\omega_c t + \theta) + n(\gamma) \end{aligned} \qquad 0 \le t \le T$$

– How to solve
  → use composite hypothesis testing

$$\Lambda_\theta(y) =$$

$$\Rightarrow \quad \Lambda(y) =$$

− How to define pdfs

* If we have m samples for data sampled at Δt

$$f_0\left(y_i\right) =$$

$$f_0\left(\underline{y}\right) =$$

* As $m \rightarrow +\infty$ and $\Delta t \rightarrow 0$

$$f_0\left(y\right) =$$

* $f_1(y) =$

* $\Lambda_\theta(y) =$

$$\Lambda\left(y\right) =$$

$$\exp\left(\frac{-A^2T}{2N_0}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{2A}{N_0}q\cos(\theta_0-\theta)\right)d\theta$$

use the fact that $\displaystyle \frac{1}{2\pi}\int_0^{2\pi}\exp\big(x\cos(\theta+\alpha)\big)d\theta = I_0(x)$

where $I_0(x)$: modified Bessel function of the first kind and zero-th order

$$\Longrightarrow \frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{2A}{N_0}q\cos(\theta_0-\theta)\right)d\theta = I_0\left(\frac{2Aq}{N_0}\right)$$

$$\Longrightarrow \Lambda(y) = \exp\left(\frac{-A^2T}{2N_0}\right)I_0\left(\frac{2Aq}{N_0}\right) \underset{<}{\overset{>}{}} \lambda_0$$

$$\Longrightarrow \qquad I_0\left(\frac{2Aq}{N_0}\right) \underset{<}{\overset{>}{}} \lambda_0\exp\left(\frac{A^2T}{2N_0}\right)$$

$$\Longrightarrow \qquad q \underset{<}{\overset{>}{}} \eta_0 \quad \text{or} \quad q^2 \underset{<}{\overset{>}{}} \eta_1$$

- ***To compute the actual decision rule:***

1) for a specified λ_0 solve for η_0 as:

$$\lambda_0 = \exp\left(\frac{-A^2T}{2N_0}\right)I_0\left(\frac{2A\eta_0}{N_0}\right)$$

2) for a NP test, threshold $\eta_0$or η_1 determined from the P_{FA}

- ***How to compute q:***

Note that expressions derived involve q$\cos(\theta_0)$ and q$\sin(\theta_0)$, one can get rid of the trigonometric expressions by computing:

$$q\cos^2(\theta_0)+ q\sin^2(\theta_0) =q^2$$

- ***Simplification for high and low SNR levels***

For low SNR: $I_0\left(\frac{2Aq}{N_0}\right) \simeq 1 + \left(\frac{Aq}{N_0}\right)^2$

$$==> \quad Ln\left(I_0\left(\frac{2Aq}{N_0}\right)\right) \simeq Ln\left(1+\left(\frac{Aq}{N_0}\right)^2\right) \simeq \left(\frac{Aq}{N_0}\right)^2$$

For high SNR: $I_0\left(\frac{2Aq}{N_0}\right) \simeq \frac{\exp\left(\frac{2Aq}{N_0}\right)}{\left(4\pi Aq/N_0\right)^{1/2}}$

$$==> \quad Ln\left(I_0\left(\frac{2Aq}{N_0}\right)\right) \simeq 2Aq/N_0 - \frac{1}{2}Ln\left(\frac{4\pi Aq}{N_0}\right) \simeq \frac{2Aq}{N_0}$$

- ***How to apply the above simplifications:***

$$I_0\left(\frac{2Aq}{N_0}\right) \underset{<}{\overset{>}{}} \lambda_0 \exp\left(\frac{A^2T}{2N_0}\right)$$

$$==> Ln\left(I_0\left(\frac{2Aq}{N_0}\right)\right) \underset{<}{\overset{>}{}} Ln(\lambda_0) + \left(\frac{A^2T}{2N_0}\right)$$

at low SNR levels:

$$\left(\frac{Aq}{N_0}\right)^2 \underset{<}{\overset{>}{}} Ln(\lambda_0) + \left(\frac{A^2T}{2N_0}\right) = \gamma$$

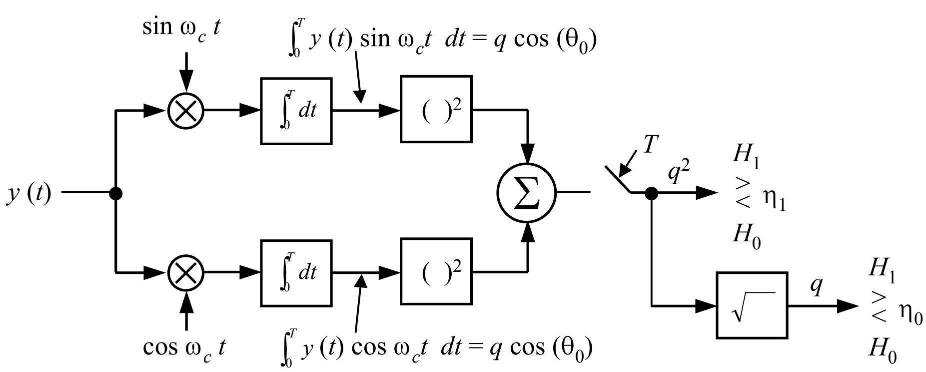
$$==> \boxed{q^2 \underset{<}{\overset{>}{}} \left(\frac{N_0}{A}\right)^2 \gamma}$$

• **How do we implement this detector ?**

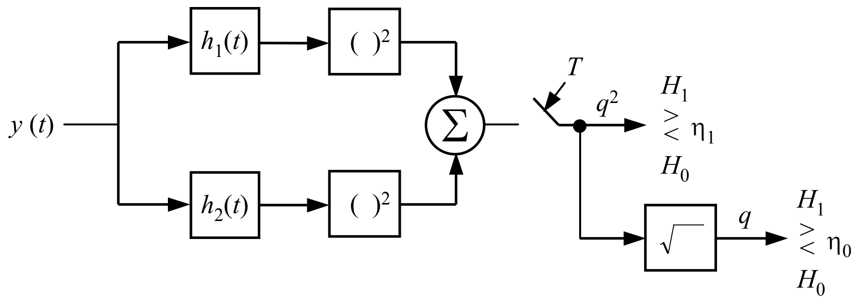
1) correlator implementation

$$\Lambda(y) = \exp\left(\frac{-A^2T}{2N_0}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{2A}{N_0}\left[\int_0^T \sin(\omega_c t)\,y(t)\,dt\cos(\theta) + \dots\right.\right.$$

$$\left.\left. + \int_0^T \cos(\omega_c t)\,y(t)\,dt\sin(\theta)\right]\right)$$



3) matched filter implementation



2) incoherent matched filter implementation

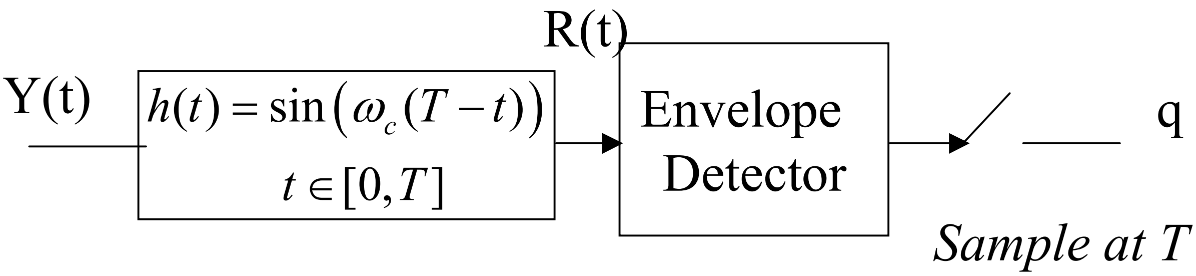
Review: Complex/real signal envelope definitions

Assume f(t) is defined as:

$$f(t) = a(t)\cos(\omega_c t + \phi(t))$$
$$= \mathrm{Re}\left[\tilde{f}(t)\exp(j\omega_c t)\right],$$
$$\text{where } \tilde{f}(t) = a(t)\exp(j\phi(t))$$
$$= a(t) + jb(t)$$

Complex envelope

Real envelope: $\left|\tilde{f}(t)\right| = \left(a^2(t) + b^2(t)\right)^{1/2}$



R(t)=

Real envelope of R(t) :

E(t)=

- **Receiver Performance**

  - Need to derive ROC curves to get an idea of the performances
  - ROC curves require to get the pdf of q

- Define: $x = \int_0^T y(t)\sin(\omega_c t)dt;\ z = \int_0^T y(t)\cos(\omega_c t)dt$

- Recall: $q = \left(x^2 + z^2\right)^{1/2}$

- Note: $x = q\cos(\alpha);\ z = q\sin(\alpha)$ may be defined where

$$\alpha = \tan^{-1}(z/x)$$

*** Under H_1:**

n(t) is Gaussian ==> x and z are Gaussian

$$f_1(x,z\,|\,\theta) = f_1(x\,|\,\theta)f_1(z\,|\,\theta), \quad why?$$

$$f_1(x,z\,|\,\theta) = \left\{\frac{1}{\sigma_x\sqrt{2\pi}}\exp\left(\frac{-1}{2\sigma_x{}^2}(x-m_x)^2\right)\right\}.$$

$$\left\{\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left(\frac{-1}{2\sigma_z{}^2}(z-m_z)^2\right)\right\}$$

- Need to compute mean and variance for H_0 and H_1

A) mean

$$E\{x \mid \theta\}_{H_1} =$$

$$E\{z \mid \theta\}_{H_1} =$$

B) Variance

$$Var\{x \mid \theta\}_{H_1} = E\left\{\left(x - E\{x \mid \theta\}_{H_1}\right)^2\right\}$$

C) Resulting pdf expression

$$f_1(x,z\,|\,\theta)=\frac{1}{\sigma^2 2\pi}\exp\left(-\frac{1}{2\sigma^2}\left[\left(x-\frac{AT}{2}\cos(\theta)\right)^2+\left(z-\frac{AT}{2}\sin(\theta)\right)^2\right]\right)$$

using the transformation: $x=q\cos(\alpha), z=q\sin(\alpha), \alpha=\tan^{-1}(z/x)$

$$f_1(q,\alpha\,|\,\theta)=\frac{q}{\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{AT}{2}\right)^2+q^2-ATq\cos(\theta-\alpha)\right]\right)$$

$$==> f_1(q\,|\,\theta)=\int_0^{2\pi} f_1(q,\alpha\,|\,\theta)d\alpha =$$

$$=\frac{q}{\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{AT}{2}\right)^2+q^2\right]\right).$$

$$\int_0^{2\pi}\frac{1}{2\pi}\exp\left(\frac{ATq\cos(\theta-\alpha)}{2\sigma^2}\right)d\alpha$$

==>

$$f_1(q\,|\,\theta)\;=\frac{q}{\sigma^2 2\pi}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{AT}{2}\right)^2+q^2\right]\right).I_0\left(\frac{ATq}{2\sigma^2}\right)$$

$$==> f_1(q)=\int_0^{2\pi}\frac{1}{2\pi}f_1(q\,|\,\theta)d\theta$$

$$\boxed{f_1(q)\;=\frac{q}{\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{AT}{2}\right)^2+q^2\right]\right).I_0\left(\frac{ATq}{2\sigma^2}\right)(1/2\pi\times 2\pi)}$$

- **Pdf under H_0**

$$f_1(q) = \frac{q}{\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{AT}{2}\right)^2 + q^2\right]\right).I_0\left(\frac{ATq}{2\sigma^2}\right)$$

Note: Under H_0, A=0

$$f_0(q) = \frac{q}{\sigma^2}\exp\left(-\frac{q^2}{2\sigma^2}\right)$$

- $\underline{P_{fa}\text{ and }P_d}$

$$P_{fa} = \int_{\eta}^{\infty} f_0(q)dq = \int_{\eta}^{\infty} \frac{q}{\sigma^2} \exp\left(\frac{-q^2}{2\sigma^2}\right) dq$$

$$= \exp\left(-\eta^2 / 2\sigma^2\right)$$

$$P_d = \int_{\eta}^{\infty} f_1(q)dq$$

$$= \int_{\eta}^{\infty} \frac{q}{\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{AT}{2}\right)^2 + q^2\right]\right) . I_0\left(\frac{ATq}{2\sigma^2}\right) dq$$

doing the changes of variable:

$v = q/\sigma$,

$$\left(AT/2\right)^2 / 2\sigma^2 = (A^2 T/2)(T/4\sigma^2)$$

$$= E / N_0$$

$$\alpha^2 = 2E / N_0$$

$$\boxed{==> P_d = \int_{\eta/\sigma}^{\infty} v \exp\left(\frac{v^2 + \alpha^2}{-2}\right) I_0(\alpha v)\, dv}$$

- ROC Curves

(2) Signals with random amplitude and phase

– Need to average effects of all random parameters.

→ random parameters shown in $f_1(y)$ only

$$\frac{\displaystyle\int_{\varphi=(A,\theta)} f_1(y\,|\,\varphi) f_A(A) f_\theta(\theta)\, d\varphi}{f_0(y)} \underset{<}{\overset{>}{}} \lambda_0$$

Recall:

$$\Lambda(y\,|\,A) = \exp\left(\frac{-A^2 T}{2N_0}\right) I_0\left(\frac{2Aq}{N_0}\right)$$

Assume $f_A(A)$ is Rayleigh distributed

$$f_A(A) = \frac{A}{A_0^2} \exp\left(\frac{-A^2}{2A_0^2}\right) U(A)$$

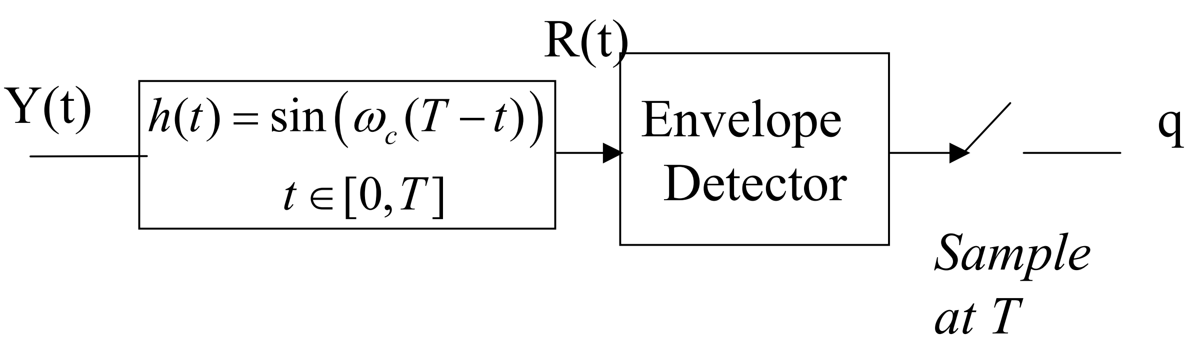
$$\Lambda(y) = \int_0^\infty \exp\left(\frac{-A^2T}{2N_0}\right) I_0\left(\frac{2Aq}{N_0}\right) f_A(A)\,dA$$

After simplifications,

$$\Lambda(y) = \frac{N_0}{N_0 + TA_0^2}\exp\left(\frac{2A_0^2q^2}{N_0(N_0+TA_0^2)}\right)$$

$$==> Ln\left(\frac{N_0}{N_0 + TA_0^2}\right) + \frac{2A_0^2q^2}{N_0(N_0+TA_0^2)} \gtrless Ln(\lambda_0)$$

$$==> q \gtrless \left[\left\{\frac{N_0(N_0+TA_0^2)}{2A_0^2q^2}\right\} Ln\left[\frac{\lambda_0(N_0+TA_0^2)}{N_0}\right]\right]^{1/2}$$



- ROC Curves

(3) Signals with random frequency and phase

- Problem occurs in radar (doppler shifting)
  in communications (time-varying channels)

- Assume: Uniform phase, and AWGN

$$\Lambda(y \mid \omega) = \exp\left(\frac{-A^2T}{2N_0}\right) I_0\left(\frac{2Aq}{N_0}\right)$$

$$\Rightarrow \Lambda(y) = \int_0^\infty \exp\left(\frac{-A^2T}{2N_0}\right) I_0\left(\frac{2Aq}{N_0}\right) f_\omega(\omega)\,d\omega$$

$$\text{where } \mathrm{q}^2 = \left[\int_0^T y(t)\sin(\omega t)\,dt\right]^2 + \ldots$$

$$+\left[\int_0^T y(t)\cos(\omega t)\,dt\right]^2$$

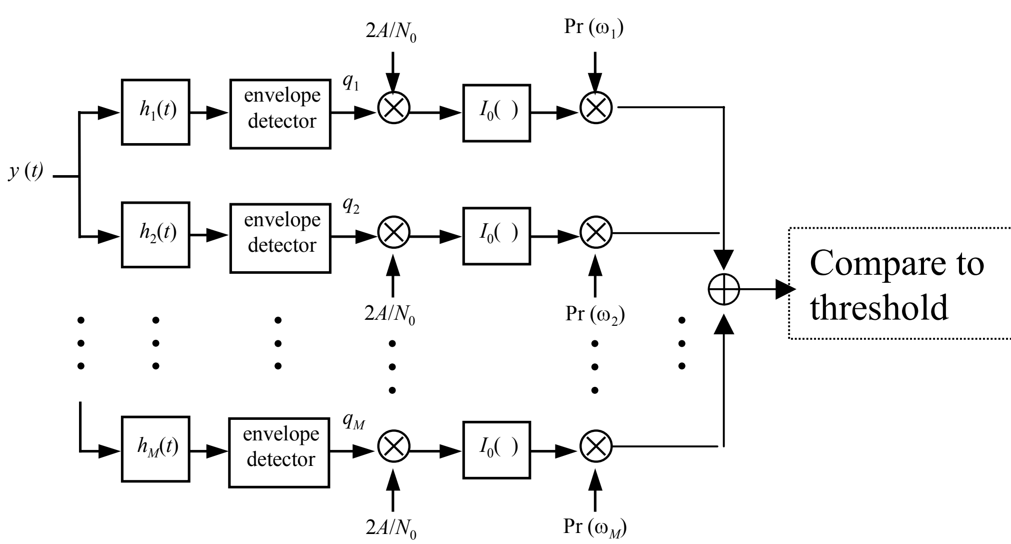
• LRT cannot be computed in a closed form solution
===> need an approximation

$$f_{\omega}(\omega) \simeq \sum_{i=1}^{M} P(\omega_i)\delta(\omega - \omega_i)$$

with $P(\omega_i) = f_{\omega}(\omega_i)\Delta\omega$

and $\omega_i = \omega_1 + i\Delta\omega, i = 1, \cdots, M$

$$\boxed{\begin{aligned}\Lambda(y) &= \int_{\omega_1}^{\omega_2} \Lambda(y \mid \omega) f_{\omega}(\omega) d\omega \\ &= \sum_{i=1}^{M} \Lambda(y \mid \omega_i) P(\omega_i)\end{aligned}}$$



• *Simplification for small SNR level*

$$\text{Recall } I_0\left(\frac{2Aq_i}{N_0}\right) \simeq 1 + \left(\frac{Aq_i}{N_0}\right)^2$$

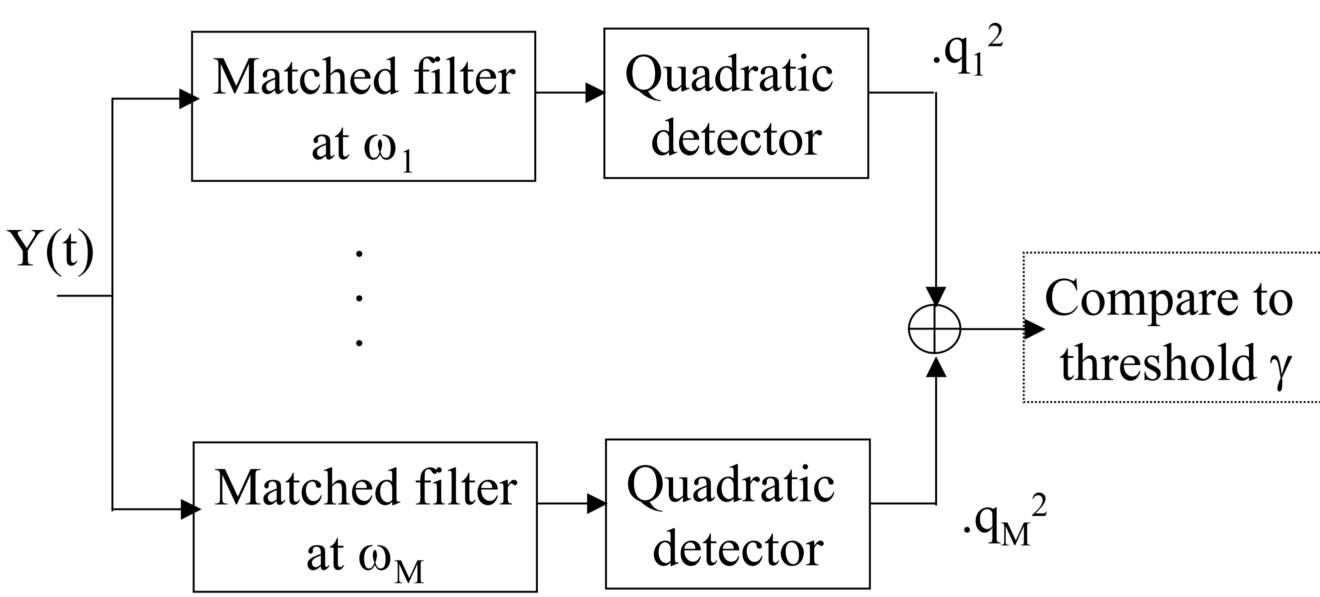
- Assume that $P(\omega_i)=1/M$

$$\Rightarrow \Lambda(y) = (1/M)\sum_{i=1}^{M} \exp\left(-A^2T/2N_0\right)\left[1+\left(Aq_i/N_0\right)^2\right]$$

- Decision rule may be expressed as:

$$\boxed{\sum_{i=1}^{M} q_i^2 \geq \gamma}$$

- Receiver



❖ Multiple Pulse Detection

- Recall: Decision made earlier based on one single pulse only.
- In practice: Multiple pulses may be available.

Ex:
- radar: pulses received sequentially
- communications: transmission over multiple channels

1) Known Radar Signal Case

Assume signals are completely known

H_0: $y_i(t) = n_i(t)$, $i=1,\ldots,M$

H_1: $y_i(t) = s_i(t) + n_i(t)$,

Return pulse at receiver

Assume received signals are independent

$$\Lambda(y) = \frac{f_1(y_1,\ldots,y_M)}{f_0(y_1,\ldots,y_M)} = \frac{f_1(y_1)f_1(y_2)\ldots f_1(y_M)}{f_0(y_1)f_0(y_2)\ldots f_0(y_M)}$$

$$= \prod_{i=1}^{M} \Lambda_i(y_i)$$

LRT for i^{th} pulse

$$f_1(y_i) = K\exp\left(-(1/N_0)\int_0^T \left[y_i(t) - s_i(t)\right]^2\right)$$

$$f_0(y_i) = K\exp\left(-(1/N_0)\int_0^T \left[y_i(t)\right]^2\right)$$

$$\Big\Downarrow$$

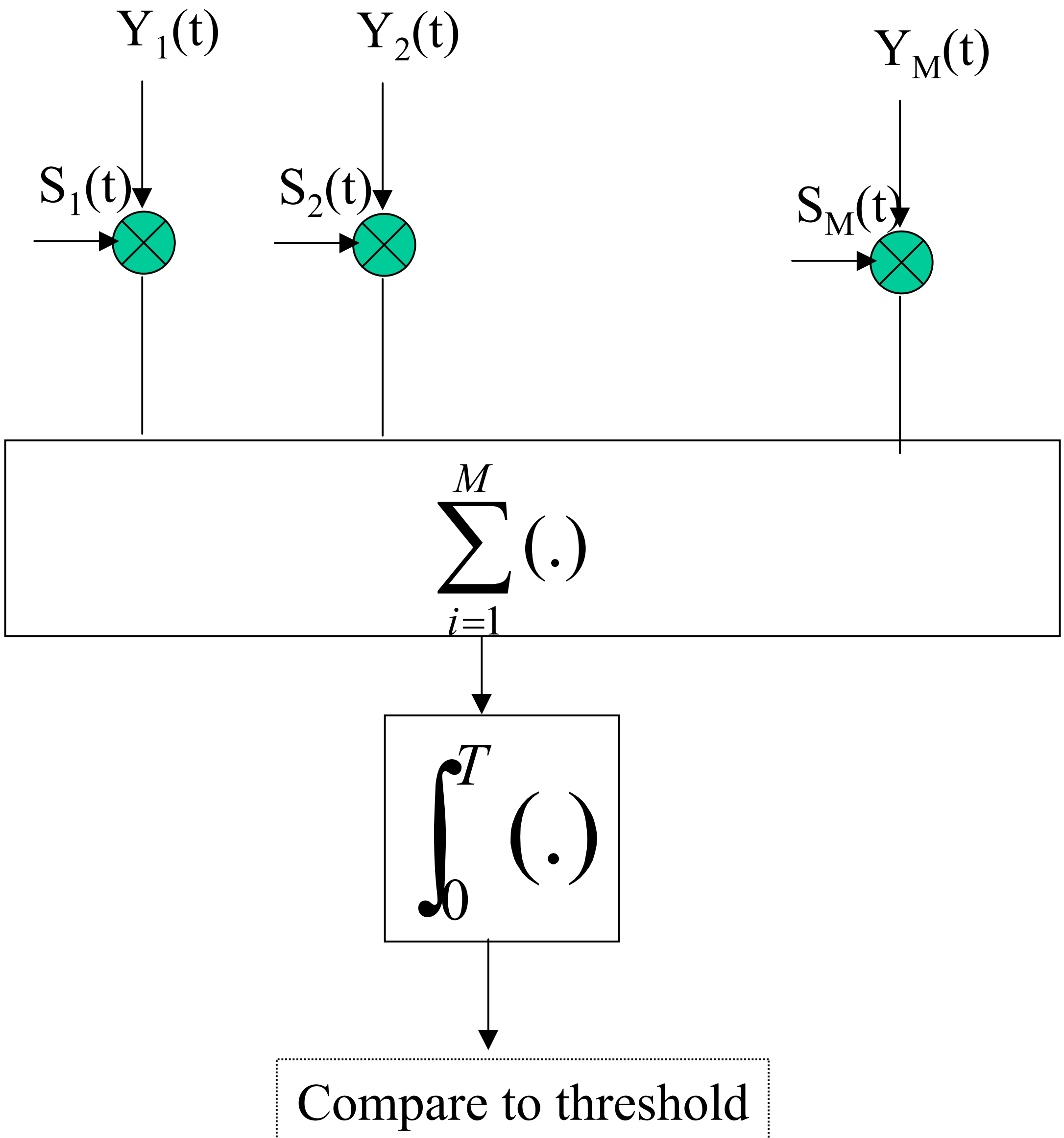
$$\Lambda_i(y_i) = \exp\left(-(1/N_0)\int_0^T s_i^{\,2}(t)dt\right)\exp\left((2/N_0)\int_0^T y_i(t)s_i(t)dt\right)$$

$$\Longrightarrow \ln[\Lambda(y)] = Ln[\prod_{i=1}^{M}\Lambda_i(y_i)]$$

$$= -(1/N_0)\sum_{i=1}^{M}E_i + (2/N_0)\sum_{i=1}^{M}\int_0^T y_i(t)s_i(t)dt \gtrless Ln(\lambda_0)$$

$$\Longrightarrow \boxed{\sum_{i=1}^{M}\int_0^T y_i(t)s_i(t)dt \gtrless \gamma \triangleq \frac{N_0}{2}Ln(\lambda_0) + \frac{1}{2}\sum_{i=1}^{M}E_i}$$

<u>Conclusion</u>: receiver structure similar to the single pulse case, except that there are now M such receivers


$Y_1(t)$
$Y_2(t)$
$Y_M(t)$
$S_1(t)$
$S_2(t)$
$S_M(t)$
$\sum_{i=1}^{M}(.)$
$\int_0^T(.)$
Compare to threshold

1) Known Signal Case - Uniform random phase

$s_i(t) = A\sin(\omega_c t + \theta_i)$

$$\Lambda_i(y_i) = \exp\left(\frac{-A^2 T}{2N_0}\right) I_0\left(\frac{2Aq_i}{N_0}\right)$$

$$\text{where } q_i^{\,2} = \left[\int_0^T y_i(t)\sin(\omega_c t)\,dt\right]^2 + \ldots$$

$$+\left[\int_0^T y_i(t)\cos(\omega_c t)\,dt\right]^2$$

$$\Lambda(y) = \frac{f_1(y_1,\ldots,y_M)}{f_0(y_1,\ldots,y_M)} = \frac{f_1(y_1)f_1(y_2)\ldots f_1(y_M)}{f_0(y_1)f_0(y_2)\ldots f_0(y_M)}$$

$$= \prod_{i=1}^{M} \Lambda_i(y_i)$$

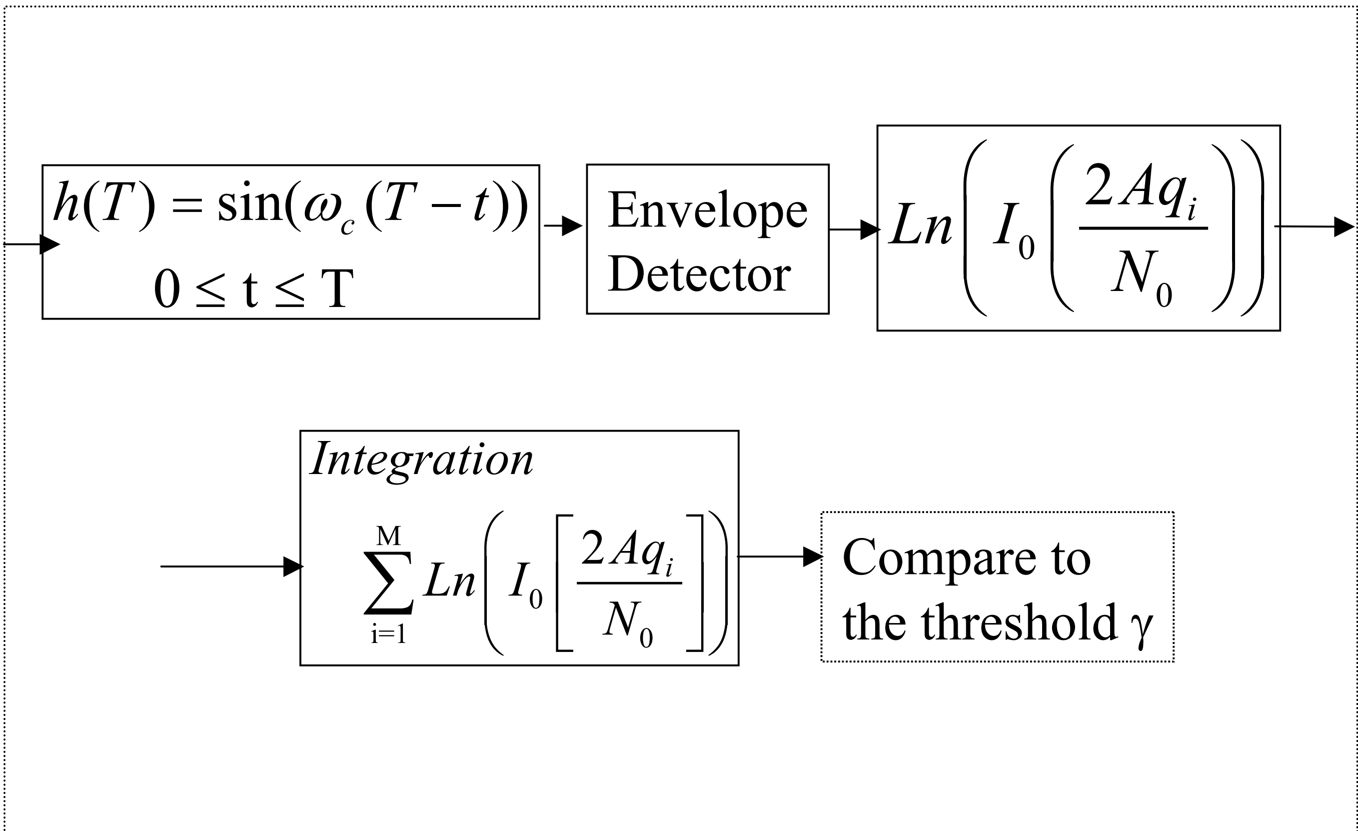
==>

$$\mathrm{Ln}[\Lambda(y)] = -\frac{MA^2T}{2N_0} + \sum_{i=1}^{M} Ln\left[I_0\left(\frac{2Aq_i}{N_0}\right)\right]$$

==>

$$\boxed{\sum_{i=1}^{M} Ln\left[I_0\left(\frac{2Aq_i}{N_0}\right)\right] \underset{<}{\overset{>}{}} \gamma \triangleq Ln(\lambda_0) + \frac{MA^2T}{2N_0}}$$

$$\sum_{i=1}^{M} I_0\left(\frac{2Aq_i}{N_0}\right) \underset{<}{\overset{>}{}} \gamma \triangleq Ln(\lambda_0) + \frac{MA^2T}{2N_0}$$



Incoherent Detector for a train of M pulses

- **Simplification for small SNR levels**

- Recall: $$I_0\left(\frac{2Aq_i}{N_0}\right) \simeq 1 + \left(\frac{Aq_i}{N_0}\right)^2$$

$$\Longrightarrow Ln\left(I_0\left(\frac{2Aq_i}{N_0}\right)\right) \simeq Ln\left(1 + \left(\frac{Aq_i}{N_0}\right)^2\right) \simeq \left(\frac{Aq_i}{N_0}\right)^2$$

- Decision rule:

$$\sum_{i=1}^{M} Ln\left\{I_0\left(\frac{2Aq_i}{N_0}\right)\right\} \gtrless \gamma \triangleq Ln(\lambda_0) + \frac{MA^2T}{2N_0}$$

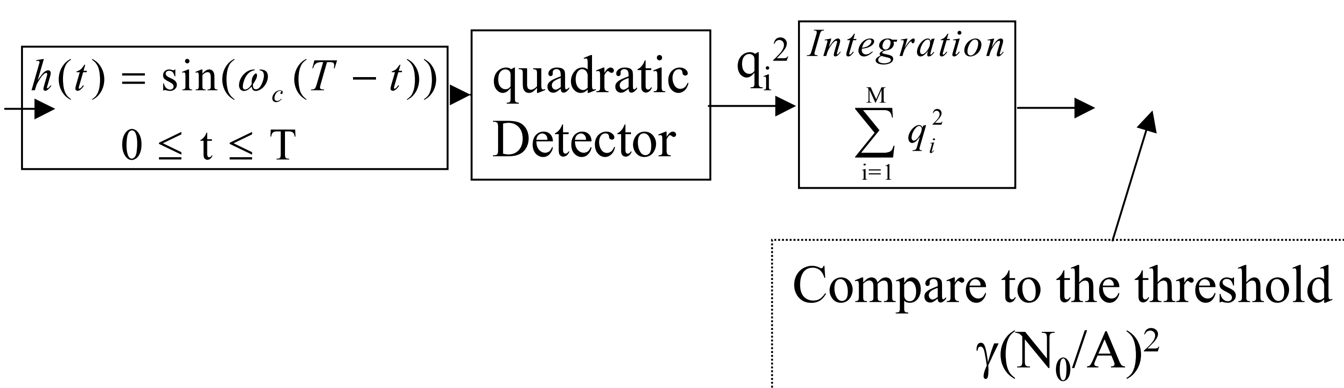
$\Longrightarrow$

$$\sum_{i=1}^{M} \left(\frac{Aq_i}{N_0}\right)^2 \gtrless \gamma$$

$\Longrightarrow$

$$\boxed{\sum_{i=1}^{M} q_i^2 \gtrless \left(\frac{N_0}{A}\right)^2 \gamma}$$

- Simplified Receiver



• Simplification for high SNR levels

- *Recall:*

$$I_0\left(\frac{2Aq_i}{N_0}\right) \simeq \frac{\exp\left(2Aq_i / N_0\right)}{\left(4\pi Aq_i / N_0\right)^{1/2}}$$

$$\Longrightarrow Ln\left(I_0\left(\frac{2Aq_i}{N_0}\right)\right) \simeq -\frac{1}{2}Ln\left(4\pi Aq_i / N_0\right) + \frac{2Aq_i}{N_0}$$

- *Decision rule:*

$$\sum_{i=1}^{M} Ln\left\{I_0\left(\frac{2Aq_i}{N_0}\right)\right\} \underset{<}{\overset{>}{}} \gamma \triangleq Ln(\lambda_0) + \frac{MA^2T}{2N_0}$$

==>

$$\sum_{i=1}^{M} \frac{2Aq_i}{N_0} \underset{<}{\overset{>}{}} \gamma$$

==>

$$\boxed{\sum_{i=1}^{M} q_i \underset{<}{\overset{>}{}} \frac{N_0}{2A}\gamma}$$

- *Simplified Receiver*