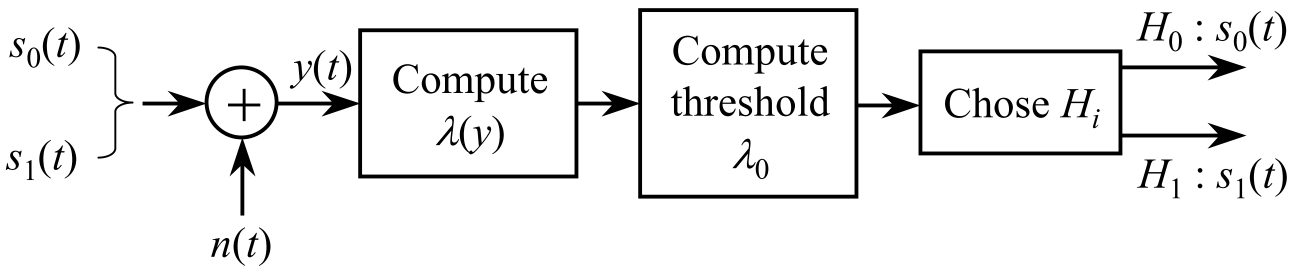
IV Detection of Dynamic Signals in White Gaussian Noise

- ❖ Binary Detection Problem
- ❖ Noise Effects
- ❖ Sampling Interval Restriction and pdf Computations
- ❖ Performance Analysis for Communication Receiver
- ❖ Examples to Communications and Radar
- ❖ Matched Filters
  - Definition
  - Matched filter used in detection
  - Extension to M-ary communications systems
- ❖ Sampled Data Approach
  - Definition
  - Independent noise samples
  - Gaussian signals
- ❖ Detection of Signals With Random Parameters
  - Signals with random phase
  - Signals with random amplitude
  - Signals with random phase & amplitude
- ❖ Multiple Pulse Detection
  - Incoherent averaging
  - Coherent averaging

Detection of Dynamic Signal in White Gaussian Noise

- Binary Detection Problem



- Assume (1)

$$
\begin{aligned}
H_0: &\quad y(t) : s_0(t) + n(t) \\
H_1: &\quad y(t) : s_1(t) + n(t)
\end{aligned}
\qquad \text{for} \quad 0 \le t \le T
$$

(2) k samples of $y(t)$ are available $\{y_1, \cdots, y_k\}$

(3) $s_0(t)$ and $s_1(t)$ have known characteristics

- Decision rule:

$\Lambda(y) =$

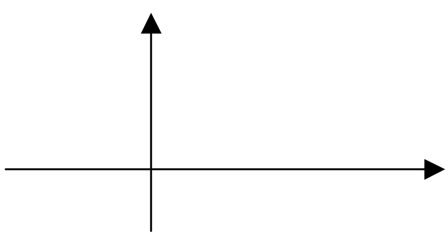
❖ Noise Effects

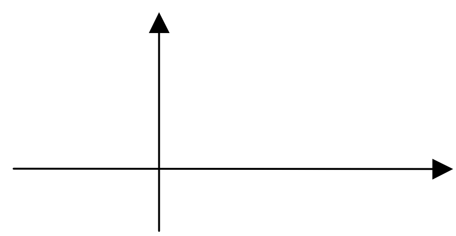
- Assume noise is bandlimited white Gaussian noise.

$$\Rightarrow \quad S_N(\omega) =$$

$$\Rightarrow \quad R_N(\tau) =$$

$$\sigma_N^2 =$$





❖ Sampling Interval Restriction and pdf Computations

- When are noise samples uncorrelated ?

- When are noise samples independent ?

- $f_0\left(y\right)=f_0\left(y_1,\cdots,y_k\right)=$

- Need mean and variance for each sample

$$E\left[y_m\right]=$$

$$\sigma_{y_m}^2 =$$

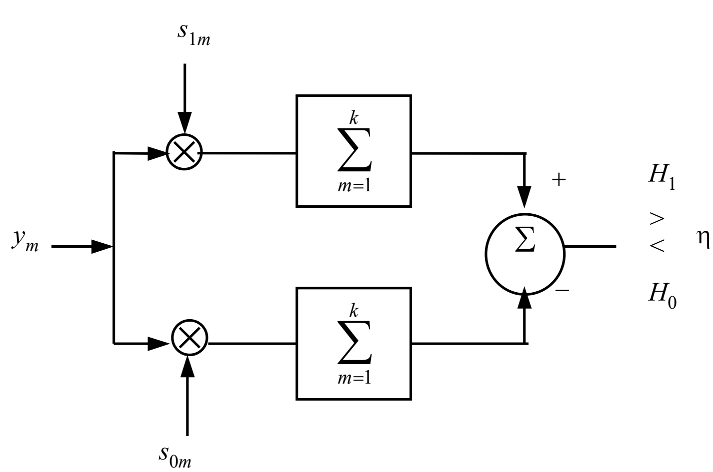
- $f_0\left(y\right)=$

- $f_1\left(y\right)=$

- $\Lambda\left(y\right)=\dfrac{f_1\left(y\right)}{f_0\left(y\right)}$

$\Rightarrow$

Digital correlator implemented as:



- Limiting case as $\Omega \rightarrow +\infty$

❖ Performance Analysis

(as a communication receiver)

- Assume $P_0 = P_1 = \dfrac{1}{2}$

  $C_{00} = C_{11} = 0; \quad C_{10} = C_{01} = 1$

- Decision rule becomes

$$G = \int_0^T y(t)\left[s_1(t) - s_0(t)\right]dt + \frac{1}{2}\int_0^T \left(s_0^2(t) - s_1^2(t)\right)dt \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

- Notes:

  (1) $y(t)$ is Gaussian $\Rightarrow$ G is Gaussian
  (normal pdf property: scaled and integrated normal pdf is also normal)

  (2) only needs $E[G]$ and σ_G^2 to define G exactly for each hypothesis

Conclusions:

- The error performance depends on 3 parameters:
  1) average signal energy
  2) level of noise spectral density
  3) time cross-correlation between signals
- Performance is independent of particular signal used
- As $(1 - p)\, E/N_0 \nearrow \quad \Rightarrow \quad Pe \searrow$
- For fixed E/N_0, optimum system obtained when $p = -1$ (i.e., when $s_0(t) = -s_1(t)$)

  → called "ideal binary communication system"

Example: Assume we have a coherent phase shift keying (CPSK) system.

$$H_0: \quad s_0(t) = A\sin\omega_0 t \qquad\qquad 0 \le t \le T$$
$$H_1: \quad s_1(t) = -A\sin\omega_0 t$$

$y(t) = s_i(t) + n(t)$ $\qquad$ $n(t)$ white Gaussian noise

$$N\left(0, \frac{N_0}{2}\right)$$

Compute the probability of error P_e.

Example: Assume we have a coherent frequency shift keying (CFSK) system.

$$\begin{aligned} H_0: \quad s_0(t) &= A\sin\omega_0 t \\ H_1: \quad s_1(t) &= A\sin\omega_1 t \end{aligned} \qquad 0 \le t \le T$$

$$y(t) = s_i(t) + n(t), i = 0,1, \qquad n(t) \sim N\left(0, \frac{N_0}{2}\right)$$

Assume ω_0 and ω_1 are chosen so that

$$\begin{cases} \omega_1 - \omega_0 = \dfrac{n\pi}{T} \\ \omega_1 + \omega_0 = \dfrac{m\pi}{T} \end{cases}$$

Above constraints on frequencies insures that
ρ =0
Proof:

Compute the probability of error P_e

Compute the probability of error P_e

Example: Assume we have a OOK keying scheme.

$$\begin{aligned} H_0: &\quad s_0(t) = 0 \\ H_1: &\quad s_1(t) = B\cos\omega_1 t \end{aligned} \qquad 0 \le t \le T$$

$$y(t) = s_i(t) + n(t), i = 0,1, \qquad n(t) \sim N\left(0, \frac{N_0}{2}\right)$$

Compute the probability of error P_e

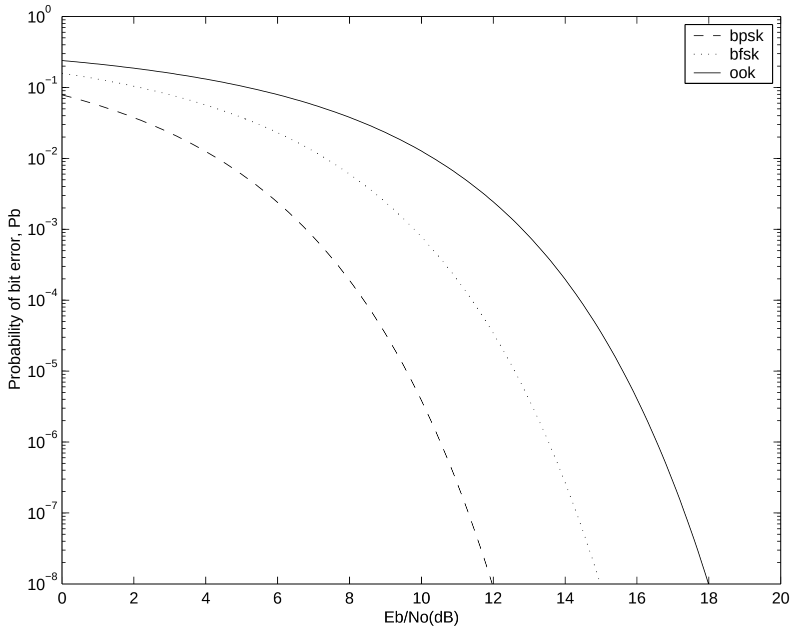
Communication schemes comparison

$$\varepsilon = 1/2 \int_0^T \left(s_0^2(t) + s_1^2(t) \right) dt$$

$$\rho = \frac{1}{\varepsilon} \int_0^T s_0(t) s_1(t) dt$$

$$P_{FA} = Q\left(\sqrt{\frac{\varepsilon(1-\rho)}{N_0}} \right); \quad P_e = P_{FA}$$

Scheme	H_0 & H_1	P_e
CPSK	$s_0(t) = A\sin(\omega_0 t)$ $s_1(t) = -A\sin(\omega_0 t)$	$\varepsilon = \frac{A^2 T}{2}; \quad Q\left(\sqrt{\frac{2\varepsilon}{N_0}} \right)$
CFSK	$s_0(t) = A\sin(\omega_0 t)$ $s_1(t) = A\sin(\omega_1 t)$	$\varepsilon = \frac{A^2 T}{2}; \quad Q\left(\sqrt{\frac{\varepsilon}{N_0}} \right)$
OOK	$s_0(t) = 0$ $s_1(t) = B\cos(\omega_1 t)$	$\varepsilon = \frac{B^2 T}{4}; \quad Q\left(\sqrt{\frac{\varepsilon_1}{N_0}} \right)$



bpsk
bfsk
ook
Probability of bit error, Pb
Eb/No(dB)

Example: Application to radar.

$$H_0: \quad y(t) = n(t) \qquad \text{(no target)}$$

$$0 \le t \le T$$

$$H_1: \quad y(t) = A\cos(\omega_1 t) + n(t) \qquad \text{(target)}$$

Recall earlier detection scheme

$$H_0: \quad y(t) = s_0(t) + n(t)$$
$$H_1: \quad y(t) = s_1(t) + n(t)$$

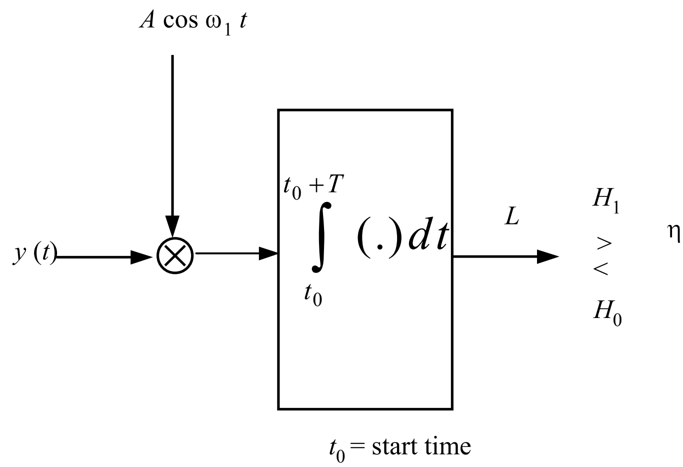
leads to

$$G = \int_0^T y(t)s_1(t)\,dt - \int_0^T y(t)s_0(t)\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}Ln(\lambda) + \frac{1}{2}\int_0^T \left(s_1^2(t) - s_0^2(t)\right)dt$$

$\Rightarrow \quad G =$

Statistics of G needed to set-up the test

- Correlator set-up



❖ Matched Filters

- What is a matched filter ?

  linear filter designed to detect the presence of a signal embedded in zero-mean noise

- What is the criterion used to design a matched filter ?

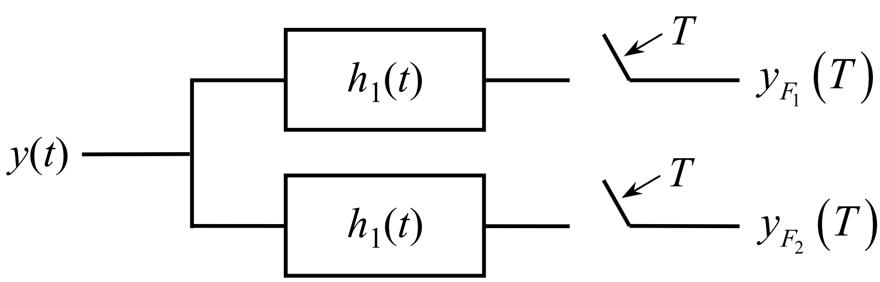
- How is the matched filter used in detection ?
  (2 signal case)

  – Assume $H_0: \quad y(t) = s_0(t) + n(t)$

    $H_1: \quad y(t) = s_1(t) + n(t)$

    $n(t)$ Gaussian noise $\quad N\left(0, \sigma_n^2\right)$
    uncorrelated with signal

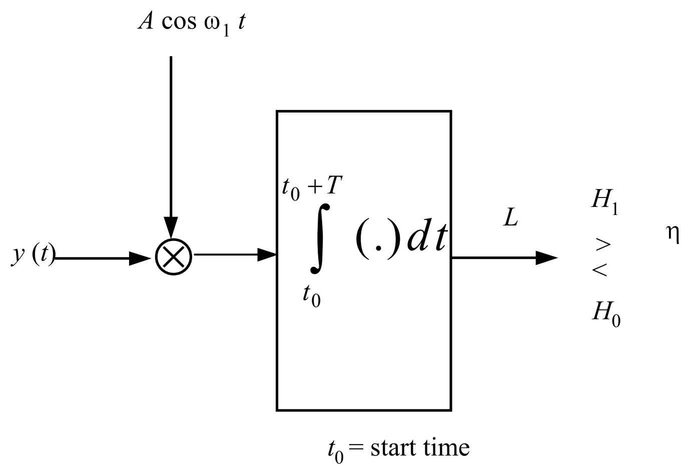
  – Receiver set-up



Assume $\quad h_1(t) =$

$h_2(t) =$

Recall correlator receiver set-up



Matched filter set-up can be used instead

- Extension to M-ary communications systems.

Assume we have M possible signals orthogonal over the interval T and with equal energy.

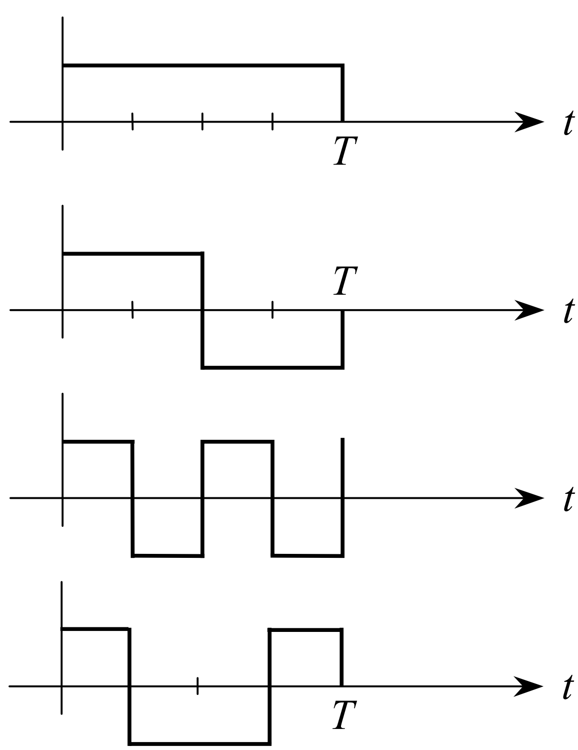
$$H_1: \quad y(t) = s_1(t) + n(t)$$
$$H_2: \quad y(t) = s_2(t) + n(t)$$

$$H_M: \quad y(t) = s_M(t) + n(t)$$

Note: $s_i(t)$, $s_j(t)$ orthogonal

$$\Rightarrow \quad \int_0^T s_i(t)\, s_j(t) = 0 \qquad i \neq j$$

Example of Orthogonal Signals



❖ Sampled Data Approach

- Goal: Apply previous results to sampled data
- Consider problem of detecting signal in noise (M samples)

$$\begin{aligned} H_1 &: \quad y(k) = s(k) + n(k) \\ H_0 &: \quad y(k) = n(k) \end{aligned} \qquad k = 1, \cdots, M$$

Assume signal characteristics are known

– noise is zero mean Gaussian with covariance R_n

$$f_1(\underline{y}) =$$

$$f_0(\underline{y}) =$$

$$\Lambda(\underline{y}) =$$

$$\ln\left[\Lambda(\underline{y})\right] =$$

- Special case: noise samples are independent and with same variance.

$$\ln\left[\Lambda\left(\underline{y}\right)\right]=$$