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II. Introduction to Signal Processing

I. Sampling

- $x(t) \longrightarrow \boxed{A/D} \xrightarrow{x(nT)} \boxed{\text{re-label}} \longrightarrow x(n)$

- Nyquist theorem: Any real signal $x(t)$ with maximum frequency $f_{\max}$ can be uniquely represented by its sampled version $x(nT)$ if the sampling frequency f_s is defined as:

$$f_s \geq 2f_{\max}$$

II. Discrete-Time Fourier Transform (DTFT)

$x(n)$ discrete sequence

$X(e^{j\omega}) =$

x(n)=

- DTFT properties

  - Recall:

(1) $X(e^{j\omega})$ periodic with period

(2) *x(t)* periodic can be expanded in terms of *exp(jkω₀t)*

(3) $x(t) = e^{j\omega_0 t} \;\; \rightarrow \;\; X(\omega) =$

- <u>As a result:</u> we should expect the DTFT of $e^{jk\omega n}$ to be periodic replicas of $X(\omega)$

Property:

$$x(n) = e^{jk\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Examples:

(1) $$x(n)=\begin{cases}1 & |n|\le N_1\\ 0 & ow\end{cases}$$

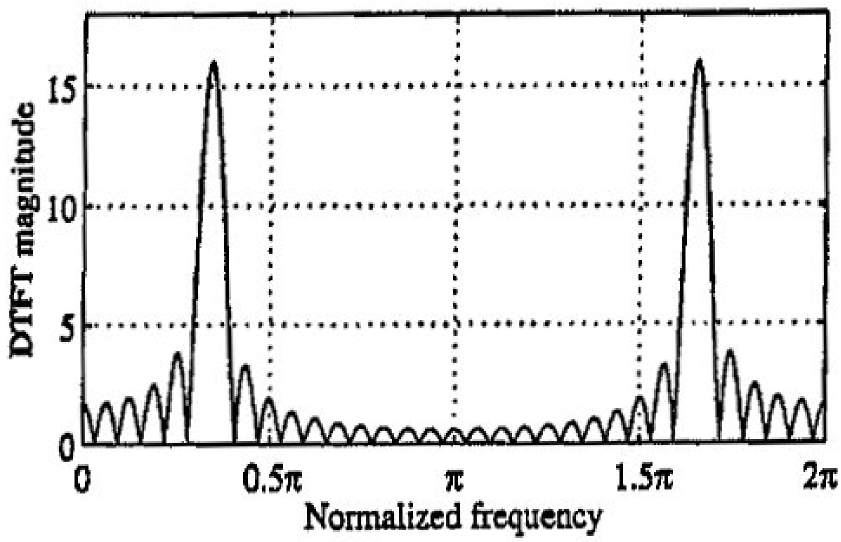
Compute and plot $X(e^{j\omega})$

(2) $x(n) = \cos\left(\frac{\pi}{3} n\right)$

Compute and plot $X(e^{j\omega})$

(3) $x[n] = \cos(\omega_0 n) \qquad 0 \le n \le N-1 \qquad X(e^{j\omega}) = ?$

$x[n] = \cos(\omega_0 n T_s)$



- <u>Example</u>:

$N = 32 \qquad \omega_0 = 0.34\pi \;\text{ rad/s} \qquad T_s = 1$

$(f_s = 1/T_s = 1 \text{ Hz})$

III. Applications of the DTFT to the Discrete Fourier Transform (DFT)

- Numerically we only have discrete values for ω on a computer
- Recall $X(e^{j\omega})$ is periodic with a period =

Definition: The R-point DFT $X[k]$ is defined as:

$X[k] =$

- Sampling interval (frequency) issues

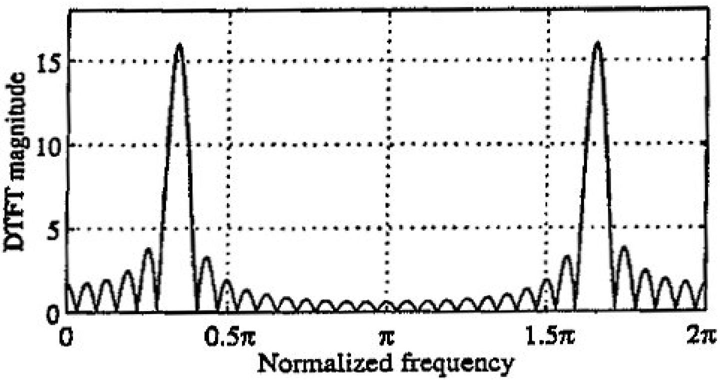
Recall $x[n]=x(nT_s)$, where T_s is the sampling interval

The sampling frequency f_s is defined as $f_s=1/T_s$

$x(t)=\cos(\omega_0 t) \implies x[n]=x(n\,T_s)=$

$=$

❖ **DTFT** $X(e^{j\omega})$ for $x[n] = \cos(\omega_0 n T_s) = \cos\left(\frac{\omega_0 n}{f_s}\right)$



$N = 32$
$f_0 = 11$ Hz; $f_s = 64$ Hz
$\omega_0 T_s =$

❖ **R-point DFT**

- The DFT of x[n] is defined as:

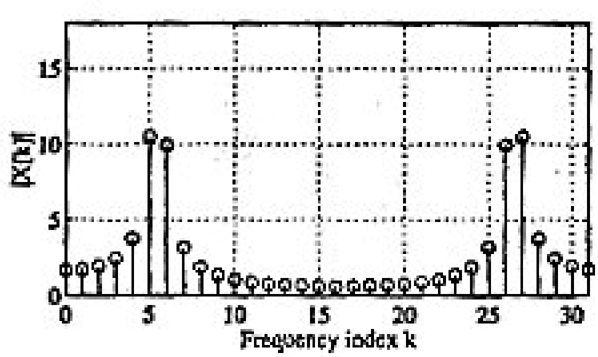
$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega = 2\pi k/R,\, R=0,\cdots,N-1}$$

Frequency bin

- X[k] is periodic with period =

- DFT Example: assume $R = 32$ points

(a)



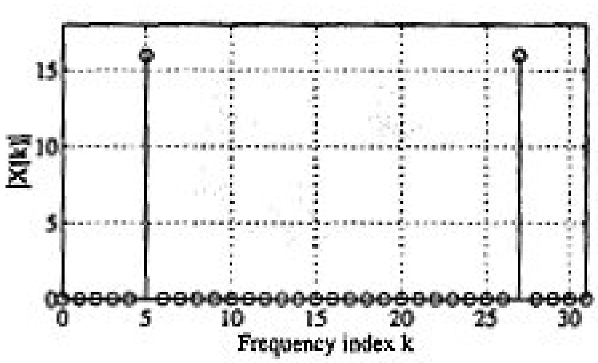
$N = 32$
$f_0 = 11$ Hz; $f_s = 64$ Hz

❖ DFT for $x[n] = \cos(\omega_0 n T_s) = \cos\left(\dfrac{\omega_0 n}{f_s}\right)$

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/R,\ R=0,\cdots,N-1}$$

Assume f_0 = 10 Hz; f_s = 64 Hz; $\omega_0 \mathrm{T_s}$ =
and R = 32 points

(b)



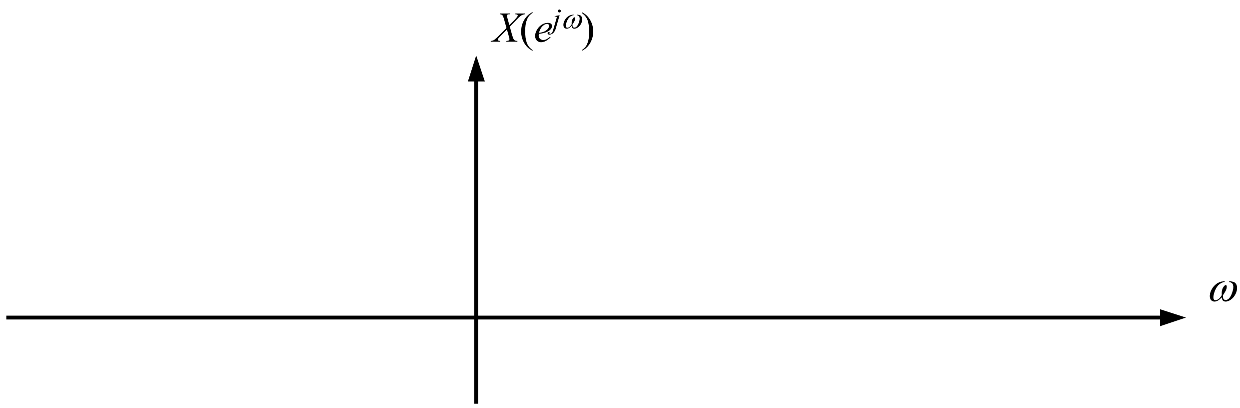
How do you explain the difference between plots (a) & (b) ?

❖ Varying the number of frequency bins *R*

Example: $x[n] = \frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right) + \cos\left(2\pi f_2 \frac{n}{f_s}\right);$

$$R = 16,\ f_s = 0.22\,Hz,\ f_2 = 0.34\,Hz,\ f_s = 1\,Hz$$

- Plot the DTFT $X(e^{j\omega})$

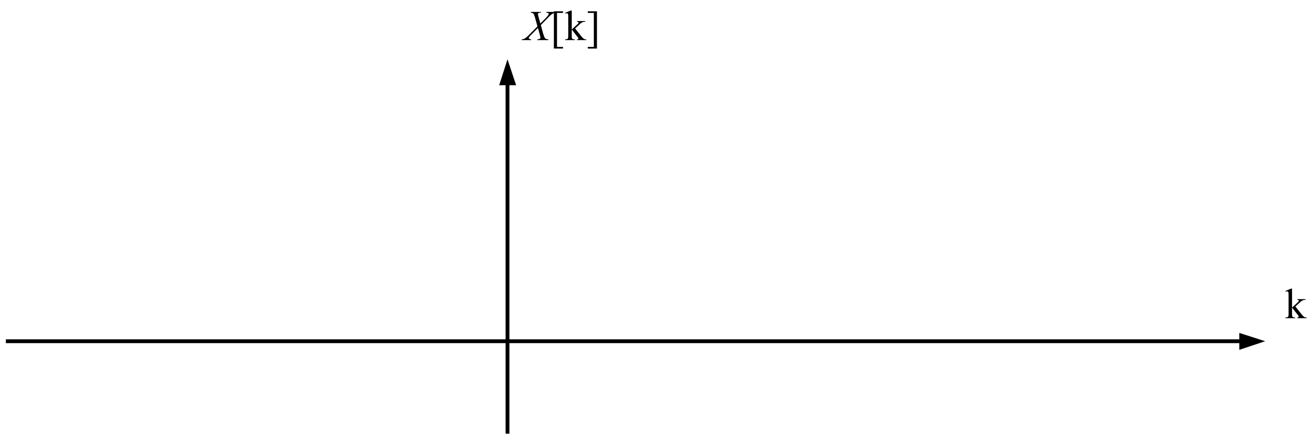


Example cont':

$$x[n] = \frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right) + \cos\left(2\pi f_2 \frac{n}{f_s}\right);$$

$$R = 16,\ \ f_s = 0.22\,Hz,\ f_2 = 0.34Hz,\ f_s = 1Hz$$

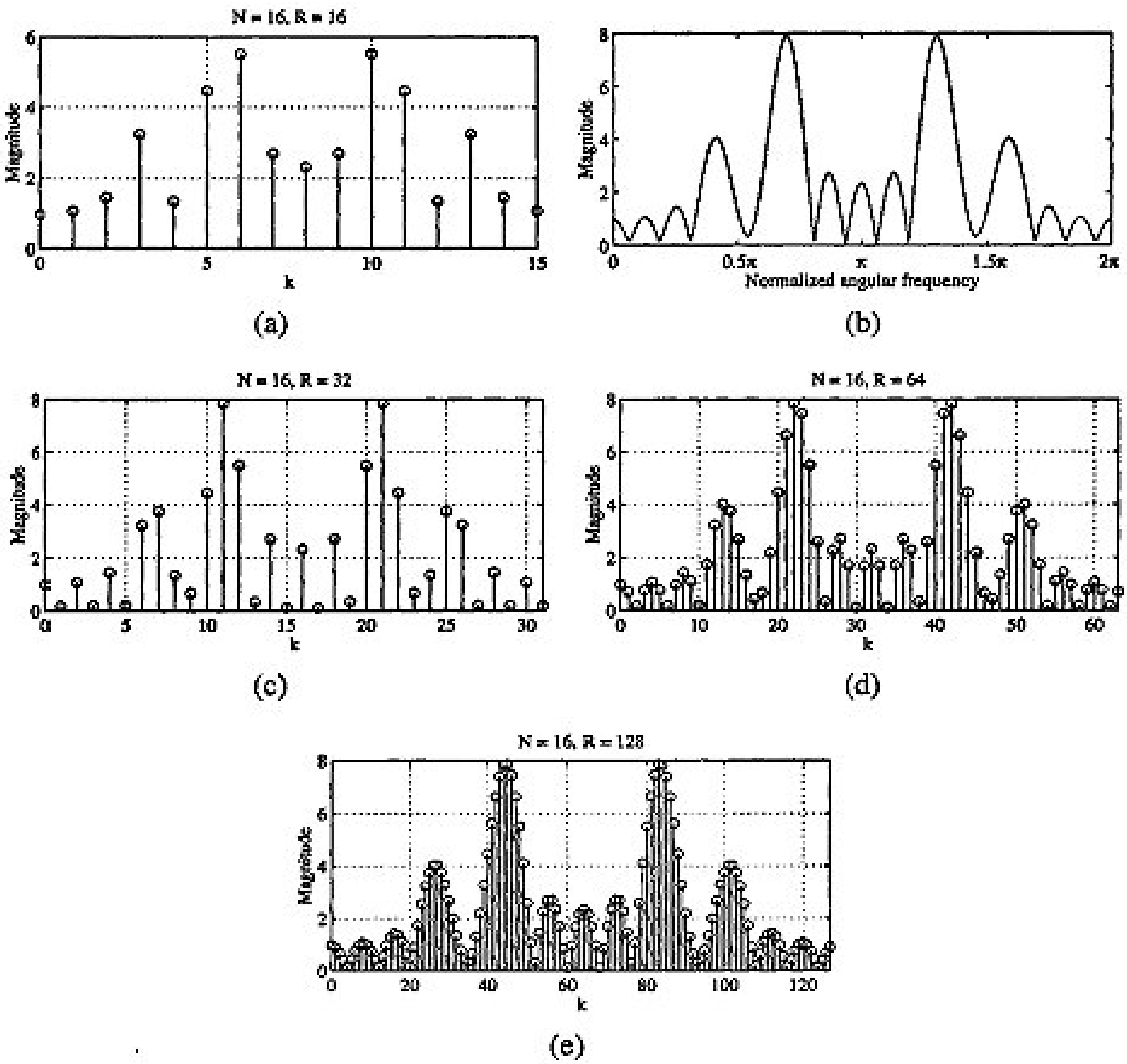
- Plot the DFT X[k]



❖**<u>Varying *R* Summary</u>;** R=[16,32,64,128]

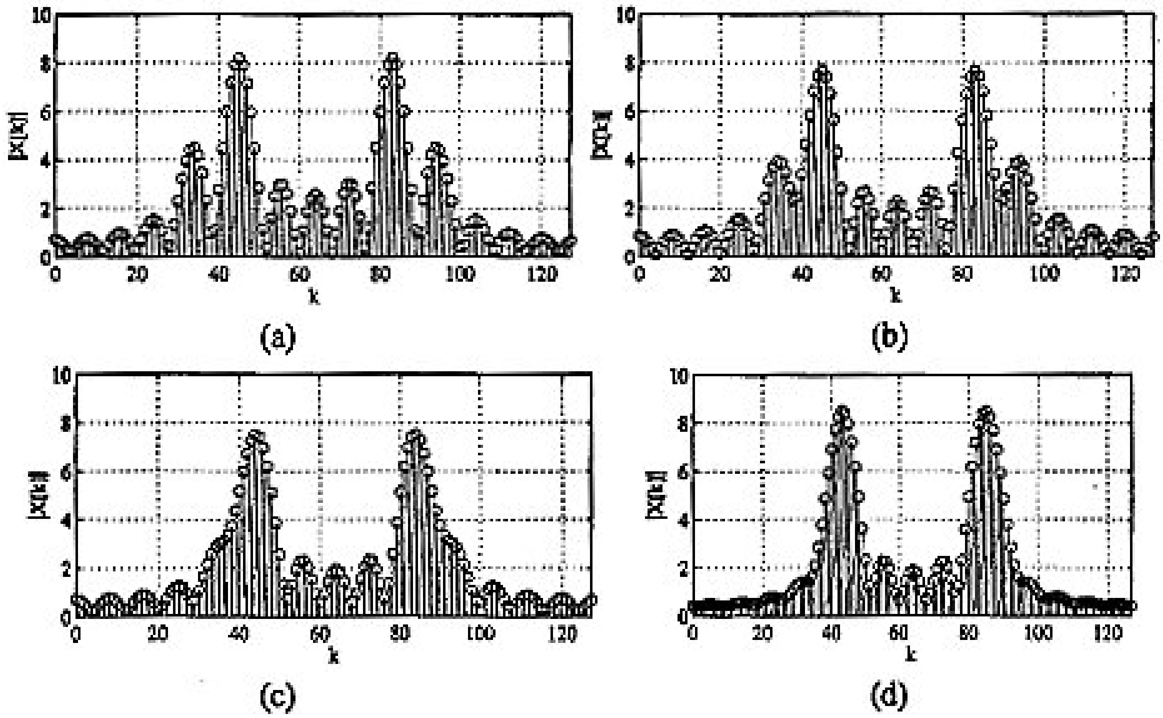
Example: $x[n]=\frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right)+\cos\left(2\pi f_2 \frac{n}{f_s}\right);$

$R=16 \qquad f_s=0.22 \qquad f_2=0.34$



❖ Resolution issues and DFT

$N = 16$; $R = 128$ (128 discrete frequencies values in [0, 2π]

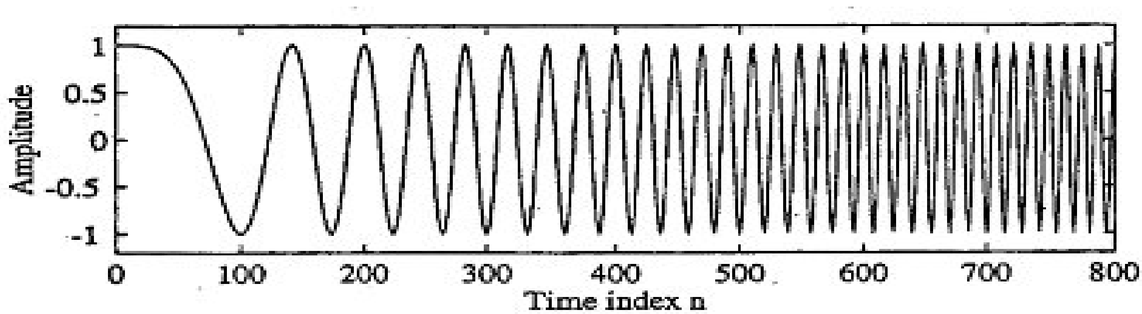


ω_2 fixed, f_s=1, f_2 = 0.34 ==>k=

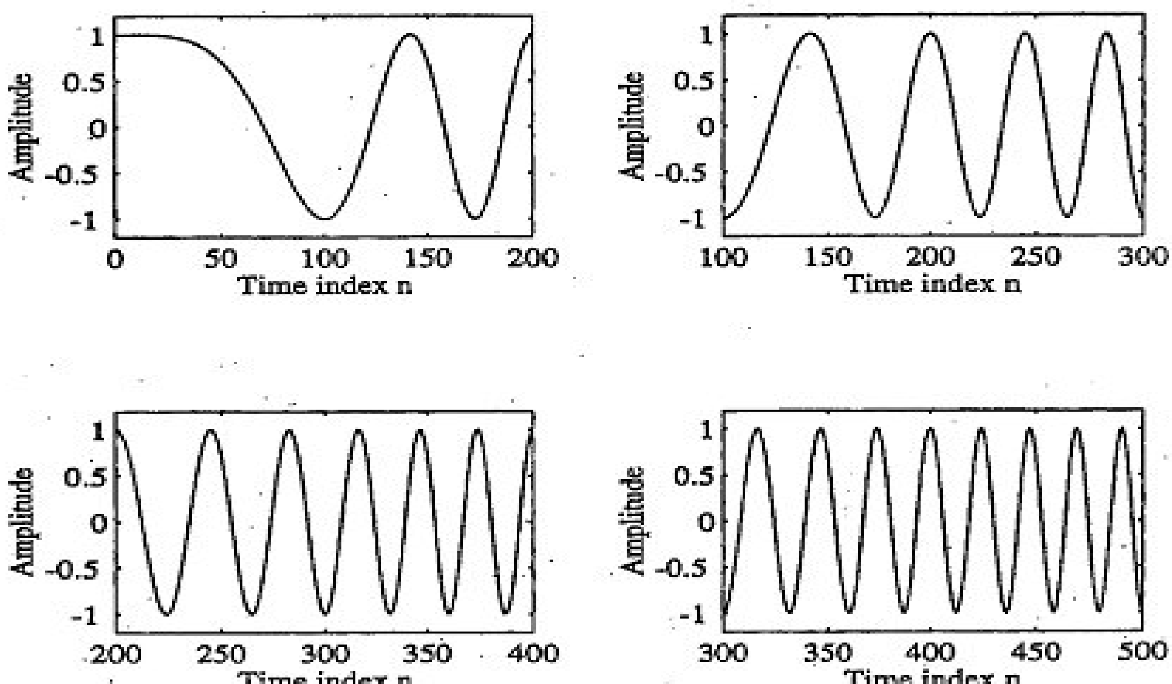
a) f_1 = 0.28 => k=
b) f_1 = 0.29 => k=
c) f_1 = 0.30 => k=
d) f_1 = 0.31 => k=

- **Comments**:

❖ Application of the DTFT to the Short-Time Fourier Transform



First 800 samples of a casual chirp signal $\cos(\omega_o n^2)$ with $\omega_o = 10\pi \times 10^{-5}$.

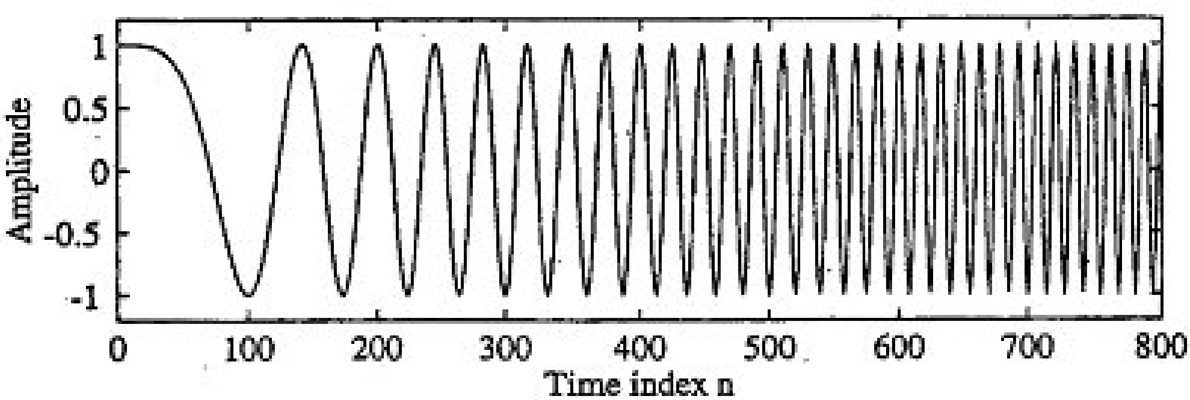


Various frames of length 200 samples extracted from the chirp signal

<u>Note</u>: Need to preserve the time-varying information of the frequency.

$\downarrow$

How to do that?



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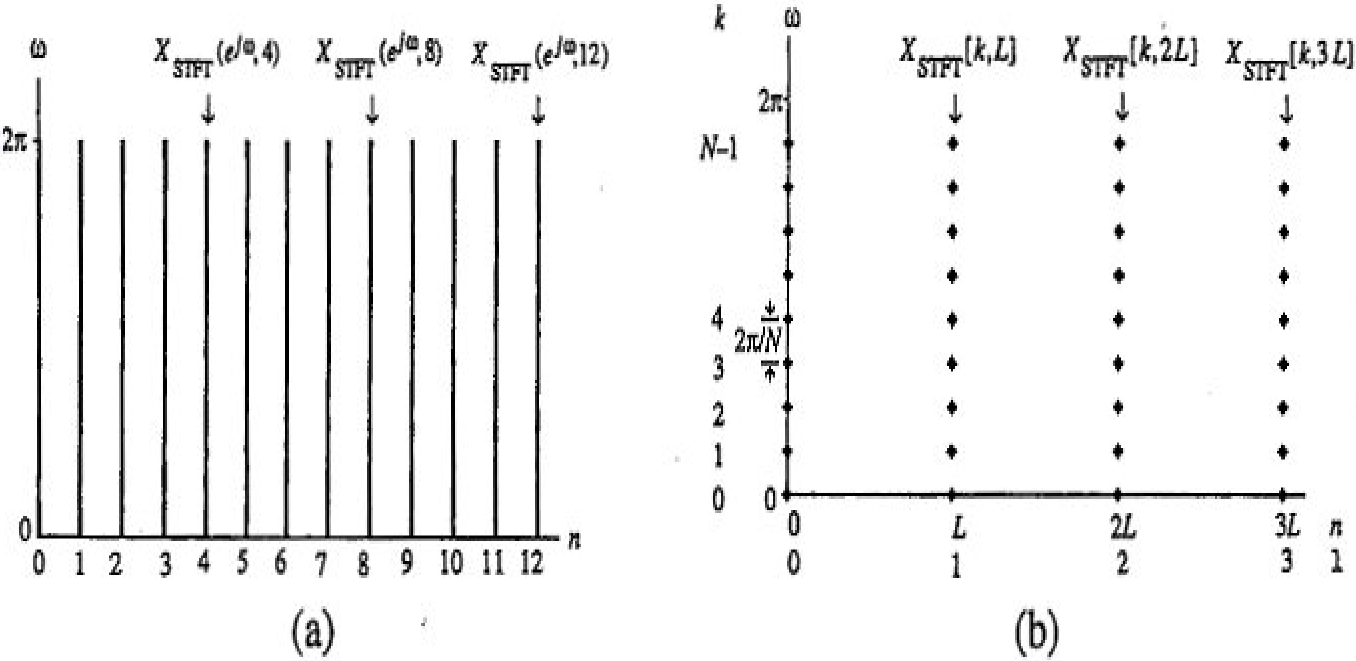

Sampling grid in the (ω,n)-plane for the sampled STFT XTFT[k,n], for N=9 & L=4

IV. FFT

- DFT of a sequence $\left\{x(n)\right\}_{n=0}^{N-1}$ defined as

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j(2\pi kn/N)}, \qquad k=0,\cdots,N-1$$

Note: (1) $X(k)$ is periodic with period ______

(2) DFT requires $0(N^2)$ operations

when $N = 2^m$ DFT requires $0(N \log_2 N)$ operations → FFT implementation

- FFT: Fast implementation of the DFT which takes advantage of $N = 2^m$ property.
  - subdivide the computation in several operations of shorter length
  - basic procedure for Radix-2 FFT

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j(2\pi kn/N)}$$

$$=$$

- **Example:** FFT implementation
  $\{x(0), \ldots x(3)\}$

V. Filtering

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

- <u>FIR filter structure</u>

$$y(n) = x(n) * h(n)$$

$$=$$

- FIR filter advantages:

- <u>IIR filter structure</u>

$$y(n) = x(n) * h(n)$$

Example:

$$x(n) \longrightarrow \boxed{\frac{1}{N}\sum_{n=0}^{N-1} x(n)} \longrightarrow y(n)$$

– Compute the filter transfer function

– -------------------- frequency response

– Plot the frequency response magnitude

– Identify the filter type

VI. Fast Correlation

- Correlation may be used to evaluate the degree of similarity between two signals.
- Recall cross-correlation is defined as:

  $R_{XY}(k) =$

- When dealing with finite length real data, $R_{XY}(k)$ is a approximated as:

  $R_{XY}(k) =$

- Cross-correlation may be implemented in the frequency domain.

VII. Periodogram

- Often desirable to estimate the power spectral density (PSD).
- PSD may be estimated as:

$$P_X(k) =$$