II. Random Processes

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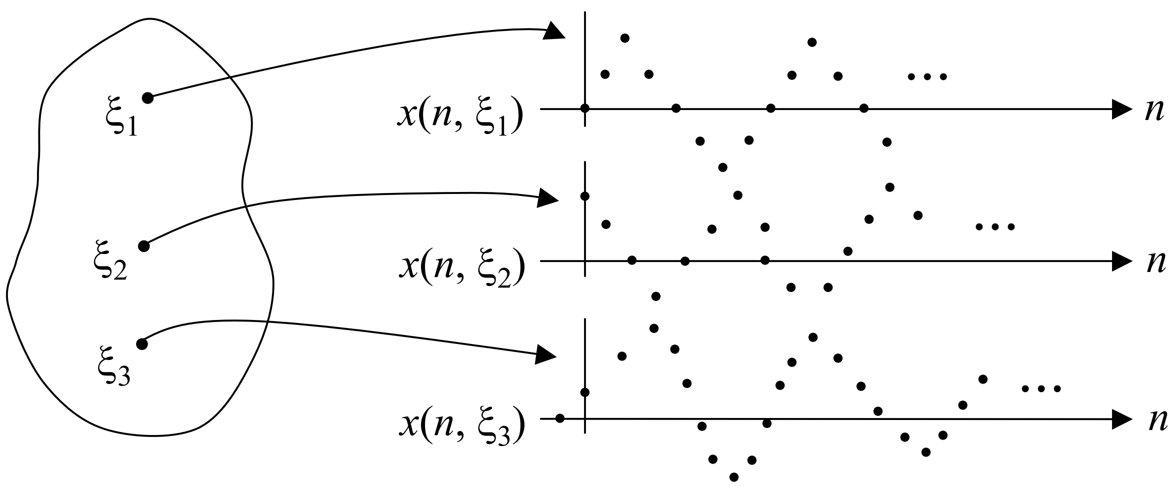
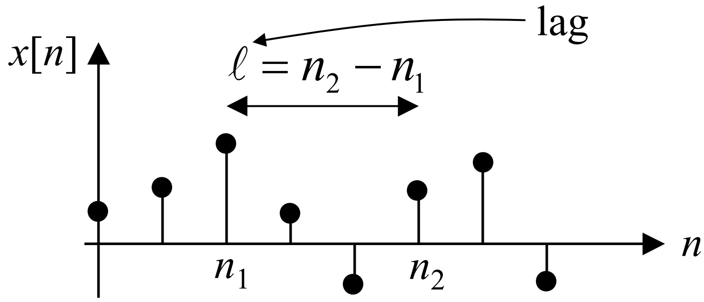
II. Random Processes

❖ <u>Random Signal/Sequence</u>:



- Consider sequence $x[n] = x(n,\xi)$ $\leftarrow$ for a fixed n x[n] is a RV
- $x[n]$: discrete random signal/ random sequence
- $x(n,\xi)$ for fixed ξ: realization of the random process (trial)

Example: $x(n,\xi)= \xi\cos(\pi n/10)$, where ξ=U[0,1].



❖ <u>Signal mean value (ensemble average):</u>

$$m_x(n) =$$

❖ <u>Signal variance:</u>

$$\sigma_x^2(n) =$$

❖ <u>Signal autocorrelation sequence:</u>

$$R_{xx}(n_1, n_2) =$$

❖ <u>Signal autocovariance sequence:</u>

$$C_{xx}(n_1, n_2) =$$

❖ <u>Signal cross-correlation sequence:</u>

$$R_{xy}(n_1,n_2)=$$

❖ <u>Signal cross-covariance sequence:</u>

$$C_{xy}(n_1,n_2)=$$

❖ <u>Normalized cross-correlation sequence:</u>

$$\rho_{xx}(n_1,n_2)=$$

❖ Statistical Characterization of Random Signals:

- Random signals are characterized by joint distribution (or density) of samples
- $F_x(x_1, x_2, \ldots, x_k, n_1, \quad n_k)$

$$= \Pr\,[x(n_1) \leq x_1, \ldots\, x(n_k) \leq x_k]$$

- F(.) is highly complex to compute - difficult to obtain in practice

❖ Stationarity:

Definition: a RP is said to be stationary if any joint density or distribution function depends only on the spacing between samples, not where in the sequence the samples occur

Example: $f_x(x_1, x_2, \ldots, x_N; n_1, \ldots, n_N)$

$$= f_x(x_1, x_2, \ldots, x_N; n_{1+k} \ldots, n_{N+k})$$

for any value of k

If $x(n)$ is stationary for all orders $N = 1, 2, \ldots$

$x(n)$ is said to be strict-sense stationary.

Example: Stationary up to order 2 $\rightarrow$ wide-sense stationary.

❖ Wide-Sense Stationarity:

➢ *Definition*: a RS $x(n)$ is called wide-sense stationary (WSS) if

(1) the mean is a constant independent of "n"

(2) the autocorrelation depends only on the distance $\ell = n_1 - n_2$ (i.e., x(n) is a seq. of uncorrelated RVs)

Consequence: the variance is a constant independent of "n"

❖ Wide-Sense Stationarity (con't):

➢ *Definition:* $x[n]$ and $y[n]$ are said to be w.s. jointly stationary if:

1) $x[n]$ and $y[n]$ are wss stationary

2) $R_{xy}[n_1, n_0] = R_{xy}[n_1 - n_0]$

➢ When $x[n]$ and $y[n]$ are w.s.j stationary:

$$R_{xy}[n_1, n_0] = R_{xy}[n_1 - n_0] =$$

$$C_{xy}[n_1, n_0] = C_{xy}[n_1 - n_0] =$$

➢ Properties:

$R_{xy}(k)=$

$C_{xy}(k)=$

❖ Example

Let $x(n)$ be a real valued process of independent variables each with mean m and variance σ^2_x.

1) Compute: $R_x(k,n)$ & $C_x(k,n)$
2) Let $y(n)$ be defined as:

$$y(n)=x(n)+x(n-1)$$

Compute: $R_y(k,n)$ & $C_y(k,n)$

❖ Signal (time) Average:

$<x[n]>=$

❖ Ergodicity:

- in many applications only one realization of a RP is available
- in general, one single member doesn't provide information about the statistics of the process
- except when process is stationary +ergodic: statistical information can be derived from one realization of RP
- ***Def:*** a RP is called ergodic if:

  all ensemble averages = all corresponding time averages

- Ergodicity in the mean:

  Def: a RP is said to be ergodic in the mean if:

- Ergodicity in correlation:

  Def: a RP is said to be ergodic in correlation if:

❖ Example: Independent, Identically Distributed (I.I.D.) Random Process (RP):

A Random Process is said to be:

. an *independent process* if:
$f_x(x_1, x_2,\ldots,x_k;n_1,\ldots,n_k) = f_1(x_1;n_1)\ldots f_k(x_2;n_k)$

. if all RVs have the same pdf $f(x)$ => $x(n)$ is called I.I.D.

Note: I.I.D. processes have no memory, where a future value would depend on past values

- Mean of I.I.D. Process:

$$m_x(n) =$$

Autocovariance:

$$C_x(n_1, n_2) =$$

Autocorrelation:

$$R_x(n_1, n_2) =$$

❖ RP Example: White noise

Definition: A random sequence $w(n)$ is called a white noise process with mean m_w and variance σ^2_w iff

$$E\{w(n)\}=m_w$$
$$R_w(k)= \sigma^2_w(k)$$

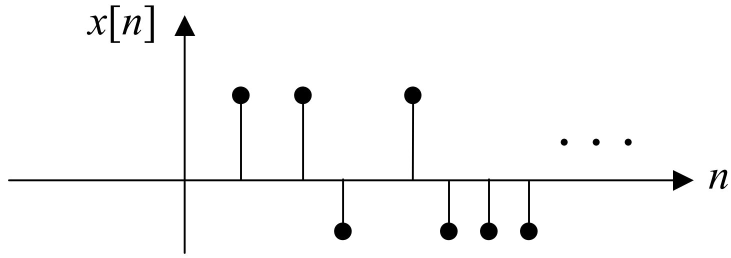
Notes:

1) all frequencies contribute the same amount (as in the case of white light, therefore the name of "white noise")

2) if the pdf of $w(n)$ is Gaussian: it is called "white Gaussian noise"

❖ RP Example: Bernoulli Process

a binary sequence – independent samples



$x[n] = 1$ with probability P
$= -1$ with probability $(1 - P)$

- for $P = 1/2$ process is called binary white noise

- Probabilistic description:
  - $Pr\,[x(0)] = 1, x[1] = 1, x[2] = -1] =$
  - is the process stationary?

Mean	Variance

❖ Application of Bernouilli process:

P[first 4 independent bits in a binary sequence are 1001] =

❖ RP Example:Random Walk

- Consider a sequence of I.I.D. RVs $\{X_i\}$
- Define $S_n = X_1 + X_2 + \ldots + X_n \qquad n = 1, 2, \ldots$

  $S_n = S_{n-1} + X_n \quad \leftarrow$ sum process

  $M_n = (1/n)\, S_n \quad \leftarrow$ arithmetic mean process
- The process S_n is called random walk when $X_i = \pm 1$ (Bernoulli RVs)
- When $P = 1/2$ (for Bernoulli process) discrete Wiener process

Properties:

S_n has independent increments in non-overlapping time intervals

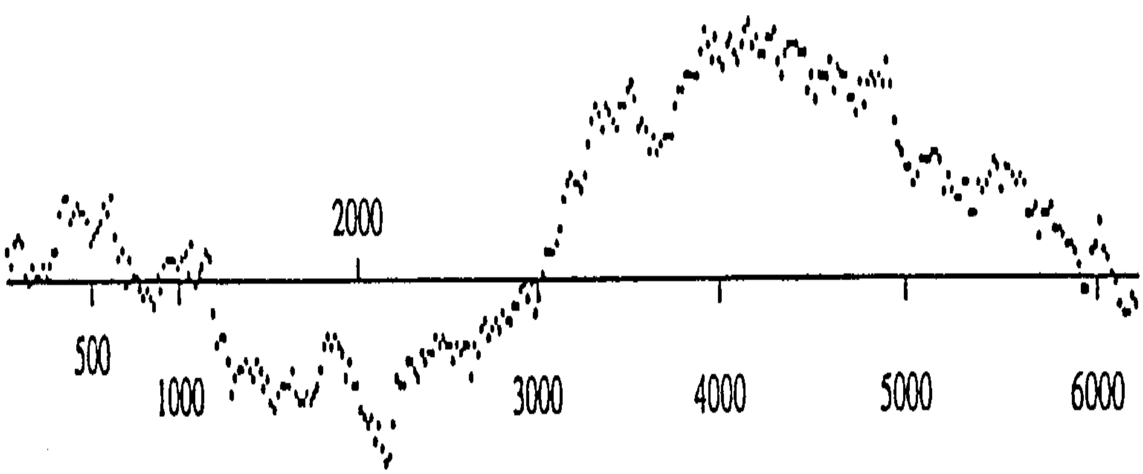
$$S_{n_1} - S_{n_2} =$$

$$S_{n_3} - S_{n_2} =$$

Random Walk: General Character

- Tends to have long runs of positive and negative values.
- Length of runs increases with increasing time, but local behavior remains the same.

Example of Random Walk:



❖ Random Process Properties

- A RP is said to be *orthogonal* if $x[n]$ is a sequence of orthogonal RVs, which means:

  $R_x(n_1, n_2) =$

- A RP is said to be *wide-sense (ws) cyclostationary* if $\exists\, N$ such that

  $m_x(n) =$
  $R_x(n_1, n_2) =$

  Example of a w.s. cyclostationary process:

Example: let $x(n)$ be a zero-mean, uncorrelated Gaussian sequence with variance $\sigma_x^2(n) = 1$

what can we say about $x(n)$?

(independent?, stationary?)

Example: let $x(n)$ be a RP generated as:

$$x(n) = \alpha x(n-1) + w(n) \quad n \geq 0$$
$$x(-1) = 0$$

with $w(n)$ a stationary RP with mean m_w and correlation sequence $R_w(\ell) = \sigma_w^2 \delta(\ell)$

a. find the mean $m_x(n)$

b. is $x(n)$ stationary?

Example: let $x(n)$ be a binary white noise process

a. compute the mean $m_x(n)$

b. compute the correlation and covariance functions

c. is $x(n)$ stationary?

Example: let $x(n)$ be a discrete Wiener process

a. compute the mean and variance

b. is the process stationary? wss?

Example: let $x(n)$ be a RP defined by a sequence of independent RVs with mean m and variance σ^2

a. Compute the mean of the RP

b. Compute the correlation of the RP

c. let $y(n) = x(n) + \frac{1}{2}x(n-1)$

- compute $R_y(n)$
- is $y(n)$ stationary?

❖ **Periodic Random Process**

- if $x[n]$ is periodic, $x[n] =$
- Density function $f_x\,[n_0]\,x[n_1]\,\ldots\,x[n_L] =$
- Mean

- Correlation

- Correlation for stationary periodic RP

Example: $x[n]=A\exp(j(\omega n+\theta)),\ \omega\sim U[0,2\pi]$
Compute $R_x(k)$ and $m_x(k)$

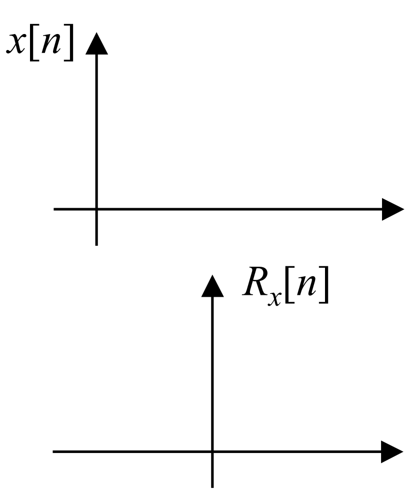
❖ <u>Correlation Function Properties</u>

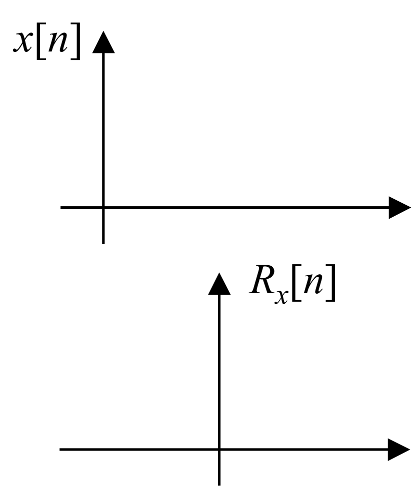
(1) <u>Conjugate symmetry</u>

(2) <u>Positive semi-definite property</u>

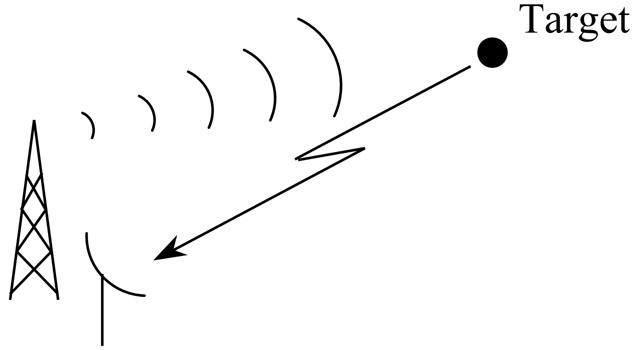
(3) <u>$R_x(k)$ max at k=0</u>

(4) High correlation   Low correlation





❖ Application to Radar Target Detection



assume $y(t) = x(t-T)$

❖ Application to Tone Detection in Noise

Property: if the process x[n] is aperiodic then

$$\lim_{k \to \infty} R_{xx}[k] = 0$$

Example:

Assume we have a sinusoidal random signal x[n] with variance 0.5 imbedded in wss white noise w[n] with variance 16 (Signal and no. The correlation sequence may be used to get information on the properties of the periodic signal

a) Compute the SNR

b) Compute the expression for the correlation of the noisy signal y[n]=x[n]+w[n].

❖ Correlation Matrix Properties for a Stationary Process

Assume $\underline{x}=[x(0),x(1)]^T$

➢ Recall: $R_x = E\,[\underline{x}\,\underline{x}^H]$

➢ Correlation Matrix for a Stationary Process

$x[n]$ stationary $\Rightarrow$

$R_x =$

- <u>Correlation Matrix for a Periodic RP</u>

Correlation Matrix Properties

Assume $\underline{x}=[x(0),\ldots,x(N-1)]^T$

(1) R_x is Hermitian

(2) R_x is positive semi definite, i.e., $\lambda(R_x)$

(3) R_x has an eigendecomposition of the form

$$R_x = U\Lambda U^H$$

where: U is a unitary eigenvector matrix

Λ is a diagonal eigenvector matrix

➢ How to compute correlation estimates

Assume $\begin{cases} x\,(t) \text{ known } t = 0 \rightarrow t = T_0 \\ x\,(t) \text{ ergodic (why?)} \end{cases}$

- For discrete data: $\underline{x}$=[x (0), , x (N)]^T

  $\hat{R}_x(k\Delta t) =$

- Quality of estimate? → find mean and variance of $\hat{R}_x(k\Delta t) =$

  (1) $E\left[\hat{R}_x(k)\right] =$

$$(2)\ \ \mathrm{Var}\left[\hat{R}_x(k)\right] \cong \frac{N}{\left[N-k\right]^2} \sum_{i=-\infty}^{\infty} \left[R_x^2(i) + R_x(i+k)\,R_x(i-k)\right]$$

when $N >> k$

Alternate Estimator: Biased Estimator

$$\widetilde{R}_x(k\Delta t) =$$

Quality of estimate:

(1) $E\left[\tilde{R}_x\left(k\Delta T\right)\right] =$

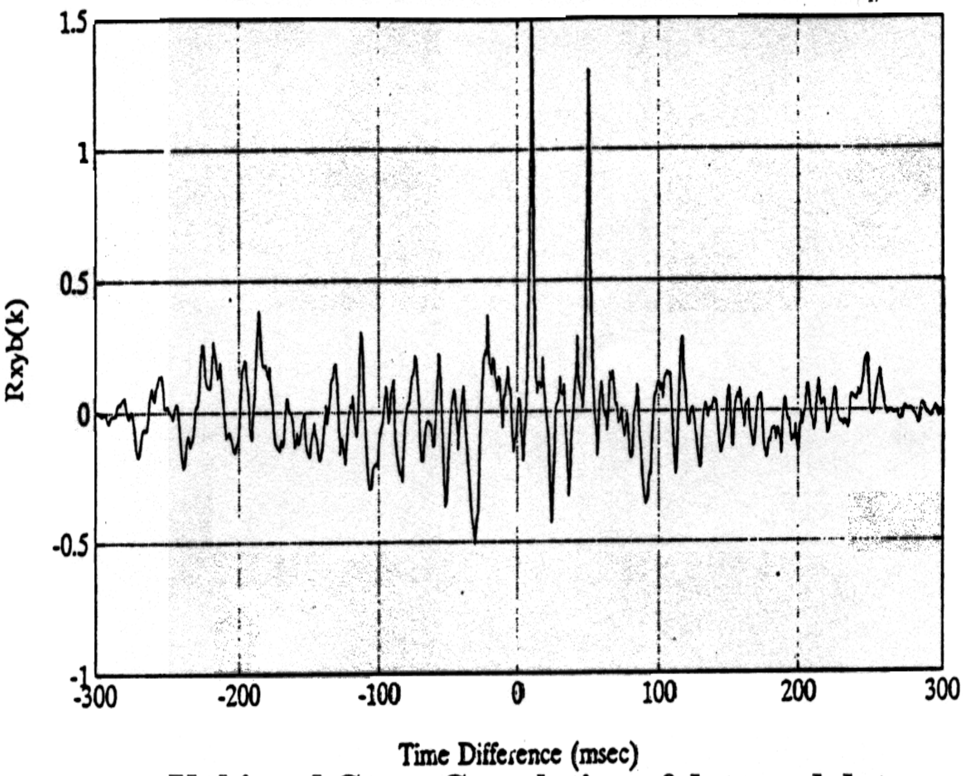
(2) $$\operatorname{Var}\left[\widetilde{R}_x(k\Delta T)\right] \cong \frac{1}{N+1}\sum_{-\infty}^{\infty} \left[R_x^2(i) + R_x(i+k)R_x(i-k)\right]$$

$$k > 0$$

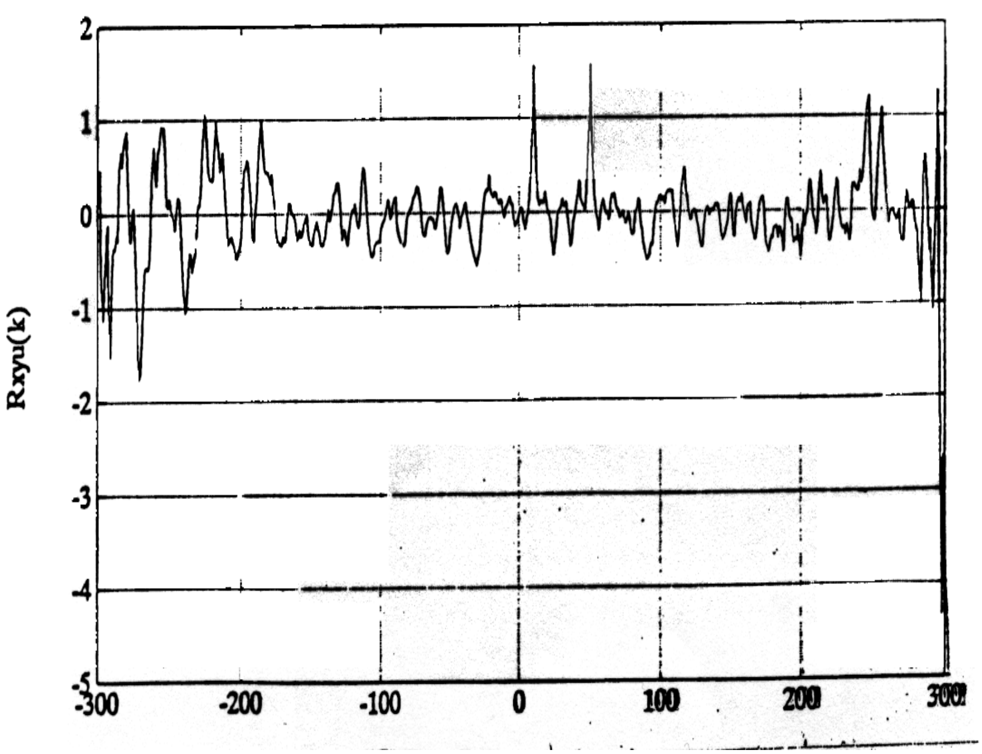
Biased/Unbiased estimator Summary

Biased Estimator	Unbiased Estimator
$\tilde{R}_x(k\Delta T)=\dfrac{1}{N+1}\sum_{i=0}^{N-k}x(i)x(i+k)$	$\hat{R}_x(k\Delta T)=\dfrac{1}{N+1-k}\sum_{i=0}^{N-k}x(i)x(i+k)$
$E\left[\tilde{R}_x(k\Delta T)\right]=\dfrac{N-k+1}{N+1}R_x(k)$	$E\left[\hat{R}(k\Delta T)\right]=R_x(k)$
$\text{Var}\left[\tilde{R}_x(k\Delta T)\right]=\dfrac{1}{N+1}\mathcal{L}(R_x)$	$\text{Var}\left[\hat{R}(k\Delta T)\right]=\dfrac{N}{\left[N-k\right]^2}\mathcal{L}(R_x)$
when $k \to N$	
$\text{Var}\left[\tilde{R}(k)\right]\to K<\infty$	$\text{Var}\left[\hat{R}(k\Delta T)\right]\nearrow +\infty$
$E\left[\tilde{R}_x(k)\right]=\dfrac{1}{N+1}R_x(k)$	$E\left[\hat{R}(k)\right]\to R_x(k)$
when $N \to +\infty$	
$\text{Var}\left[\tilde{R}_x(k)\right]\searrow$	
bias of $E\left[\tilde{R}_x(k)\right]\searrow$	

Biased Cross-Correlation of datx and daty



Unbiased Cross-Correlation of datx and daty



- <u>How to compute correlation matrix estimates</u>

❖ Frequency Domain Description of Stationary Processes

➢ Power spectral density (PSD)

$$S_x\left(e^{j\omega}\right) =$$

$$R_x\left(\ell\right) =$$

➢ Example**:** find the PSD of zero-mean w.s.s. $x[n]$ with

$$r_x\left(\ell\right) = a^{|\ell|}$$

➢ PSD Properties

Let $x[n]$ be a stationary and periodic RP, then

1) $R_x(k)$ is periodic with the same period

2) $S_x(e^{j\omega})$ is periodic with period 2π

Proof:

❖ PSD has three key properties:

(1) $P_1 = \begin{cases} \text{PSD } S_x(e^{j\omega}) \text{ is a real-valued periodic function} \\ \text{of period } 2\pi \text{ for any } x[n] \\ \text{if } x[n] \text{ is real then } S_x(e^{j\omega}) \text{ is also even} \end{cases}$

(2) $P_2 =$ the PSD $S_x(e^{j\omega})$ is non-negative definite;
I.e., $S_x(e^{j\omega}) \geq 0$ --> see page 159, text

(3) $P_3 =$ the area under $S_x(e^{j\omega})$ is non-negative and equals the energy of $x[n]$

Example: White Noise

Example: Harmonic Process

- *Definition:* a harmonic process is defined as:

$$x[n] = \sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k); \omega_k \neq 0$$

where M, $\{A_k\}$, $\{\omega_k\}$ are constants

$\{\phi_k\}$ are pairwise independent RVs uniformly distributed over $[0, 2\pi]$

- Compute $E\{x[n]\}$ and $R_x(\ell)$, $S_x(e^{j\omega})$

Example: given $R_x(\ell) = \sigma^2 \rho^{|\ell|}$ $|\rho| < 1$

compute $S_x\left(e^{j\omega}\right)$

❖ **Complex Spectral Density Function**

$$S_x(z) =$$

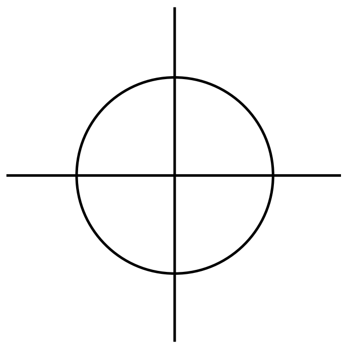
$$R_x(\ell) =$$

- Symmetry property

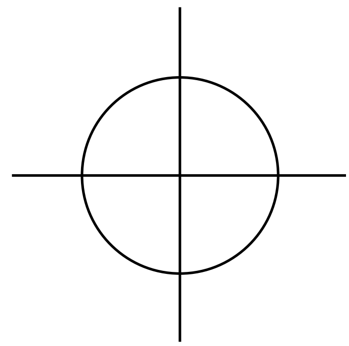
- Generic form for complex spectral density
  - General form:
  - Rational form:

❖ Properties for rational form (poles/zero location)

Complex process



Real process



❖ Region of Convergence of $S_x(z)$

Example: $R_x(\ell) = \sigma^2 \rho^{|\ell|}$ $|\rho| < 1$

compute $S_x(z)$

❖ Summary of Properties for Stationary x[n]

Definitions	
Mean	
Correlation	
Covariance	
Cross-Correlation	
Cross-Covariance	
PSD	
Cross-PSD	

Inter-relations	
$C_x(\ell) =$	$C_{xy}(\ell) =$

Properties	
Autocorrelation	PSD

Properties	
Cross-correlation	Cross-PSD

- **Example:** $x[n] = Ae^{j\omega_0 n}$ $\qquad A \quad RV$

  compute $R_x(\ell)$

  $S_x(e^{j\omega})$

❖ **Innovation Representation of Random Vectors -** the Discrete Karhunen-Loeve Transform (DKLT)

- In many practical applications, it is beneficial to represent a random sequence $\underline{x}$ with a linearly equivalent sequence $\underline{w}$ consisting of uncorrelated components (such sequence $\underline{w}$ is called the *innovation representation*).
- In such cases, each component of the uncorrelated sequence $\underline{w}$ can be viewed as adding new information to the previous components.
- Applications exist in compression, classification, etc...

➢ How to transform $\underline{x}$ into the innovation representation

Assume $\underline{x}=[x(0),\ldots x(N-1)]^T$ is zero-mean
Questions:
(1) What does it mean for $\underline{w}$ to be uncorrelated ?

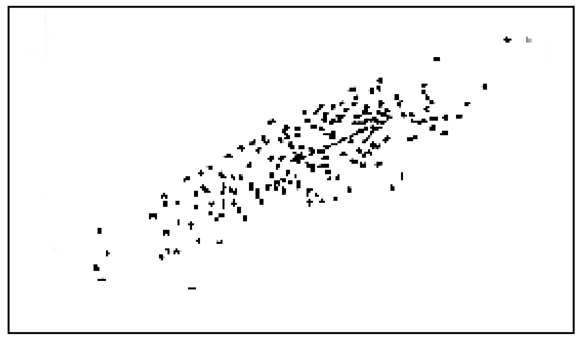
(2) What does "represent a random sequence $\underline{x}$ with a linearly equivalent sequence $\underline{w}$" mean ?

Assume $A=U^H$

➤DKLT Applications

DKLT is used in data compression (speech/image coding) as it allows for a lower dimensional representation of the data.

Example: Assume we have 200 samples of two-dimensional data of type $\underline{x}=[x_1,x_2]$.



Estimated correlation matrix is given by:

$R_x=$

Compression set-up