V. Discrete Time Fourier Transform & Discrete Fourier Transform

DTFT definition
Fourier transform for periodic signals
Examples
Applications to the Discrete Fourier Transform (DFT)
Varying the number of frequency bins R
Resolution issues and the DFT
Applications of the DTFT to the Short-time Fourier Transform

V. Discrete Time Fourier Transform & Discrete Fourier Transform

❖ **<u>Recall results for continuous signals</u>**

$x(t) =$ $\qquad\qquad\qquad$ $X(j\omega) =$

<u>D</u>iscrete <u>T</u>ime <u>F</u>ourier <u>T</u>ransform (DTFT)

$x[n]$

$X(e^{j\omega})$

called spectrum

❖ **<u>Note</u>:** $X(e^{j\omega})$ is periodic with period:

Plots from class text
Additional reference and plots for DFT section:[1] Digital signal processing, K. Mitra

$$X(e^{j\omega}) =$$

$$x[n] =$$

- <u>Example</u>:

$$x_1[n] = a^n u[n] \qquad |a| < 1$$

$$x_2[n] = a^{|n|} \qquad |a| < 1$$

$$x_3[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & \text{ow} \end{cases}$$

❖ **<u>Fourier transform for periodic signals</u>**

- Recall: $X\left(e^{j\omega}\right)=\sum_{-\infty}^{+\infty} x[n]e^{-j\omega n}$

  → DTFT periodic with period 2π

- Recall: (1) x(t) periodic can be expanded in terms of $e^{jk\omega t}$

  (2) $x(t)=e^{jk\omega t}$ ==> $X(j\omega)=2\pi\delta(\omega-\omega_0)$

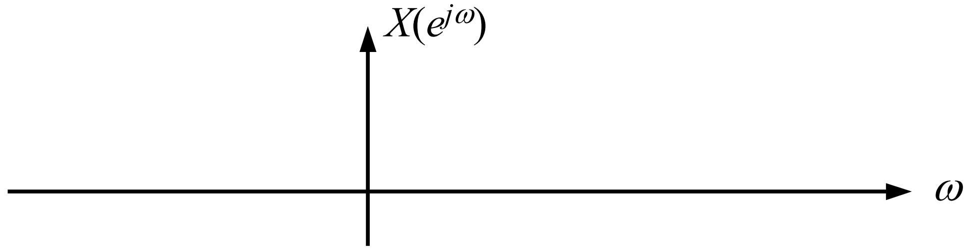
- <u>As a result</u>: we should expect the DTFT of $e^{jk\omega n}$ to be periodic replicas of $X(j\omega)=2\pi\delta(\omega-\omega_0)$

- <u>Property</u>: $x[n]=e^{j\omega_0 n}$

$$X\left(e^{j\omega}\right)=\sum_{\ell=-\infty}^{+\infty} 2\pi\delta\left(\omega-\omega_0-2\pi\ell\right)$$

- <u>Proof</u>:

- <u>Example</u>: $x[n] = \cos(\omega_0 n) \qquad \omega_0 = \pi/3$



❖ Extension

$$x[n] = a_0 + a_1 e^{j(2\pi n/N)} + a_2 e^{j(2\pi n/N)} + \cdots + a_{N-1} e^{j(N-1)(2\pi n/N)}$$

$$X(e^{j\omega}) =$$

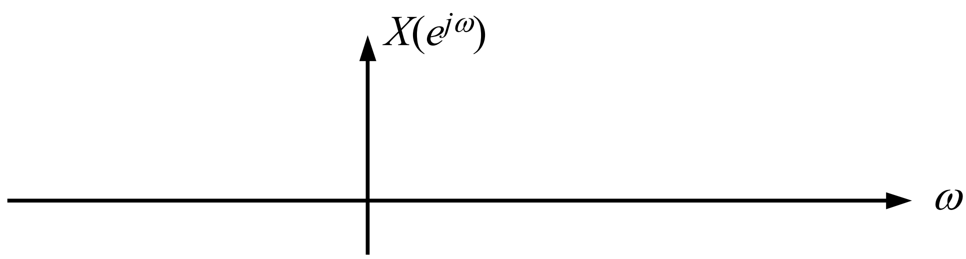


TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with
		$y[n]$	$Y(e^{j\omega})$ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$
		$x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals	$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

Note: similar properties to the FT.

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

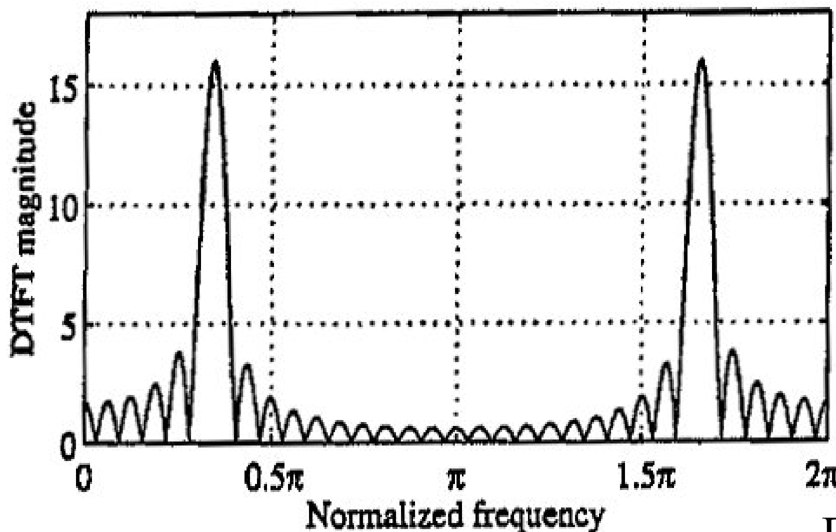
TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time				Discrete time		
	Time domain		**Frequency domain**		**Time domain**		**Frequency dom**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time		$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	duality	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	duality	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-}$ discrete frequency periodic in freque
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	duality	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency		$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time		$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j}$ continuous freque periodic in freq

- Example:

$x[n] = \cos(\omega_0 n) \quad 0 \le n \le N-1 \quad X(e^{j\omega}) = ?$

$x[n] = \cos(\omega_0 n T_s)$



Ref:[1]

- <u>Example</u>:

  $N = 32$ $\qquad \omega_0 = 0.34\pi$ rad/s $\qquad T_s = 1$

  $(f_s = 1/T_s = 1 \text{ Hz})$

x[n], y[n] aperiodic $\qquad$ *X(e^{jω}), Y(e^{jω})* periodic with period 2π

Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
Time shifting	$x[n-n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Freq. shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time reversal	$x[-n]$	$X(e^{-j\omega})$
Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

❖ Applications of the DTFT to the Discrete Fourier Transform (DFT)

- Numerically we only have discrete values for ω on a computer
- Recall $X(e^{j\omega})$ is periodic with a period =

Definition: The R-point DFT $X[k]$ is defined as:

$X[k] =$

- Sampling interval (frequency) issues

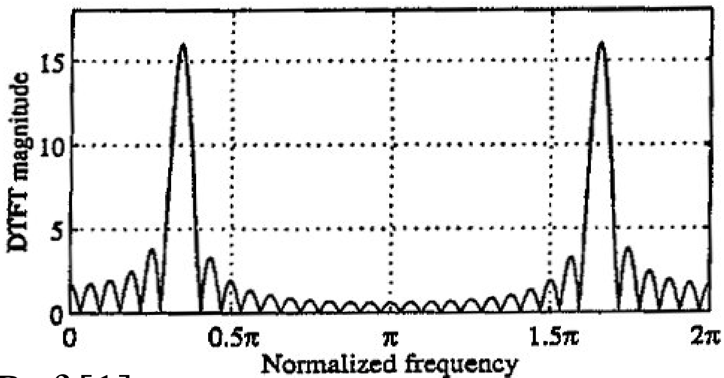
Recall $x[n]=x(nT_s)$, where T_s is the sampling interval

The sampling frequency f_s is defined as $f_s=1/T_s$

$x(t)=\cos(\omega_0 t) \Longrightarrow x[n]=x(n\,T_s)=$

$=$

❖ **DTFT** $X(e^{j\omega})$ for $x[n] = \cos(\omega_0 n T_s) = \cos\left(\frac{\omega_0 n}{f_s}\right)$



$N = 32$
$f_0 = 11$ Hz; $f_s = 64$ Hz
$\omega_0 T_s =$

Ref:[1]

❖ **R-point DFT**

- The DFT of x[n] is defined as:

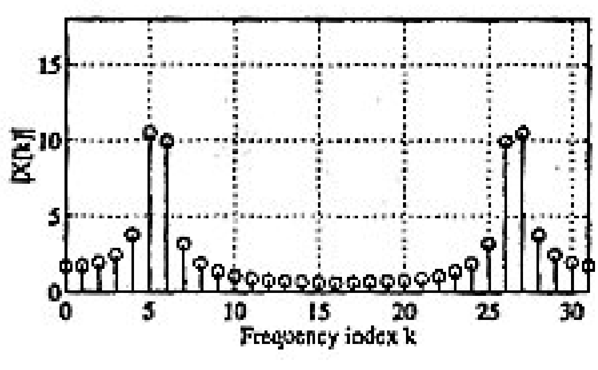
$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega = 2\pi k/R,\ k = 0, \cdots, R-1}$$

Frequency bin

- X[k] is periodic with period =

- DFT Example: assume $R = 32$ points

(a)



$N = 32$
$f_0 = 11$ Hz; $f_s = 64$ Hz

Ref:[1]

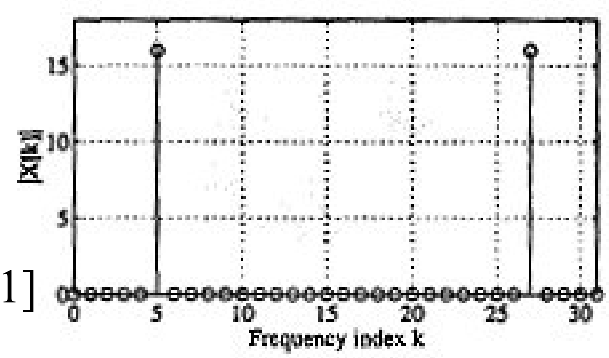
❖ DFT for $x[n] = \cos(\omega_0 n T_s) = \cos\left(\frac{\omega_0 n}{f_s}\right)$

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/R,\ k=0,\cdots,R-1}$$

Assume $f_0 = 10$ Hz; $f_s = 64$ Hz; $\omega_0 T_s =$
and $R = 32$ points

(b)

Ref:[1]



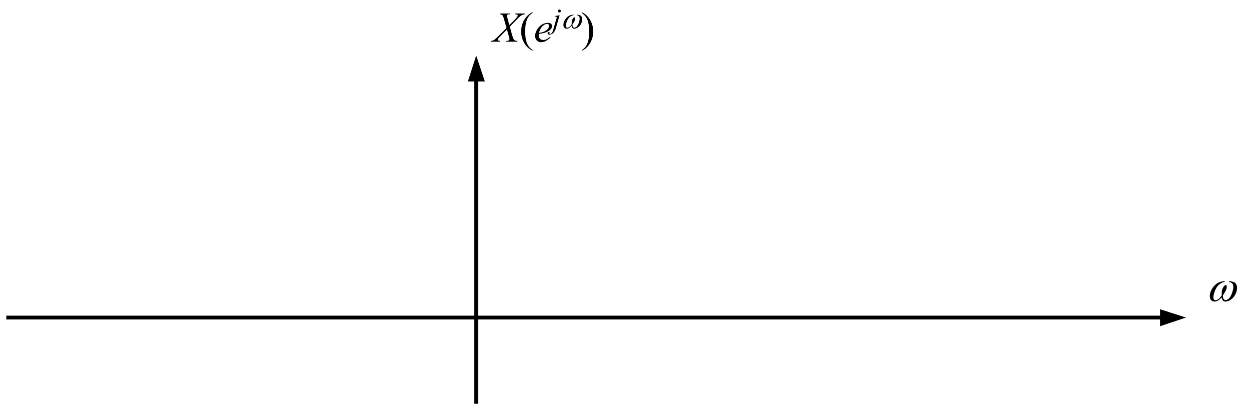
How do you explain the difference between plots (a) & (b) ?

❖ Varying the number of frequency bins *R*

Example: $x[n] = \frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right) + \cos\left(2\pi f_2 \frac{n}{f_s}\right);$

$$R = 16,\ \ f_s = 0.22\,Hz,\ f_2 = 0.34Hz,\ f_s = 1Hz$$

- Plot the DTFT $X(e^{j\omega})$

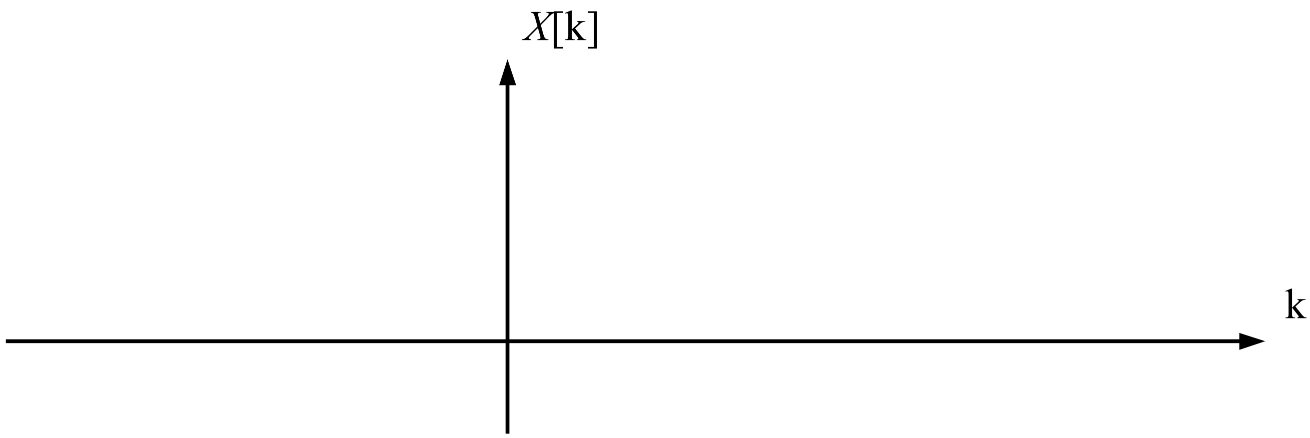


Example cont’:

$$x\left[n\right]=\frac{1}{2}\cos\left(2\pi f_s\frac{n}{f_s}\right)+\cos\left(2\pi f_2\frac{n}{f_s}\right);$$

$$R=16,\ \ f_s=0.22\,Hz,\ f_2=0.34Hz,\ f_s=1Hz$$

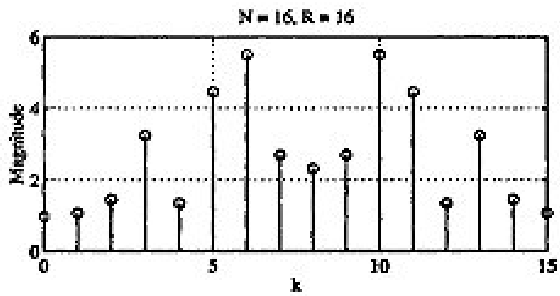
- Plot the DFT X[k]



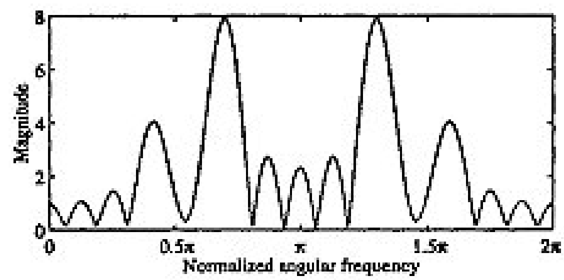
❖**<u>Varying *R* Summary</u>;** R=[16,32,64,128]

Example: $$x[n]=\frac{1}{2}\cos\left(2\pi f_s \frac{n}{f_s}\right)+\cos\left(2\pi f_2 \frac{n}{f_s}\right);$$

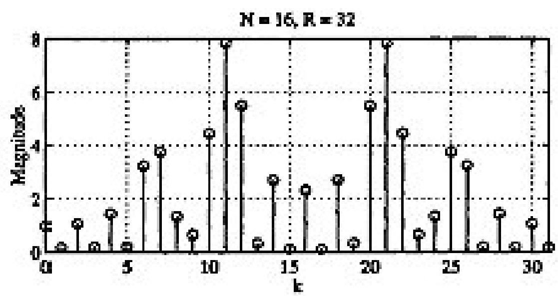
$$R=16 \qquad f_s=0.22 \qquad f_2=0.34$$



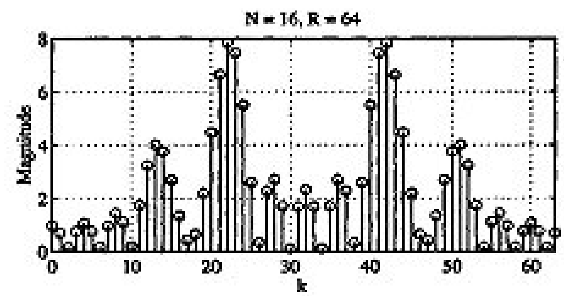
(a)



(b)

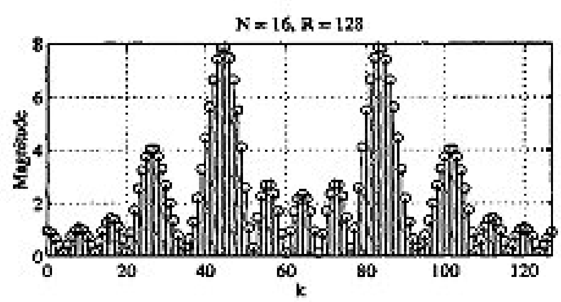


(c)



(d)

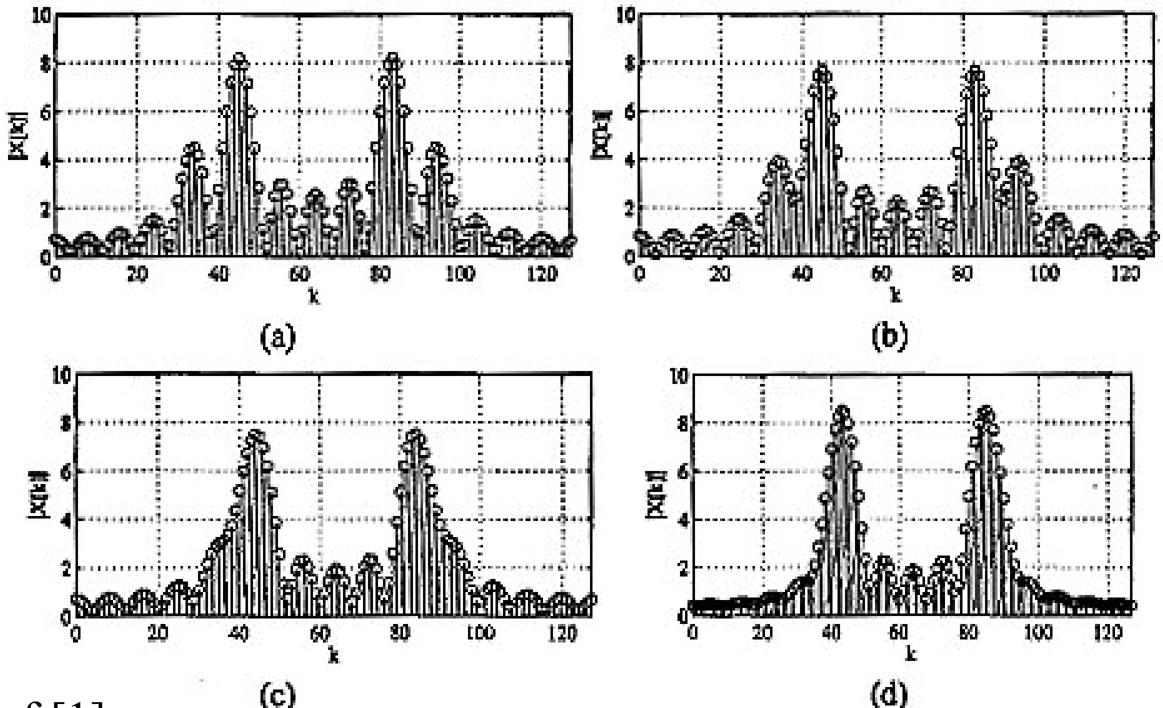
Ref:[1]



(e)

❖ <u>Resolution issues and DFT</u>

$N = 16$; $R = 128$ (128 discrete frequencies values in [0, 2π]



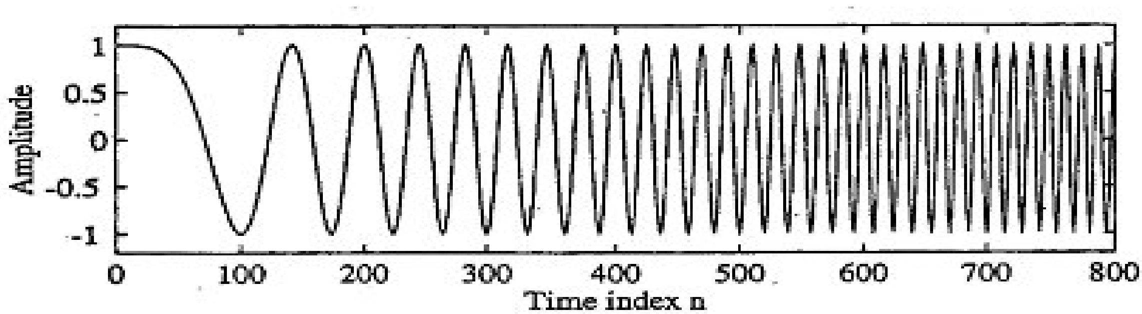
Ref:[1]

ω_2 fixed, f_s=1, $f_2 = 0.34$ ==>k=

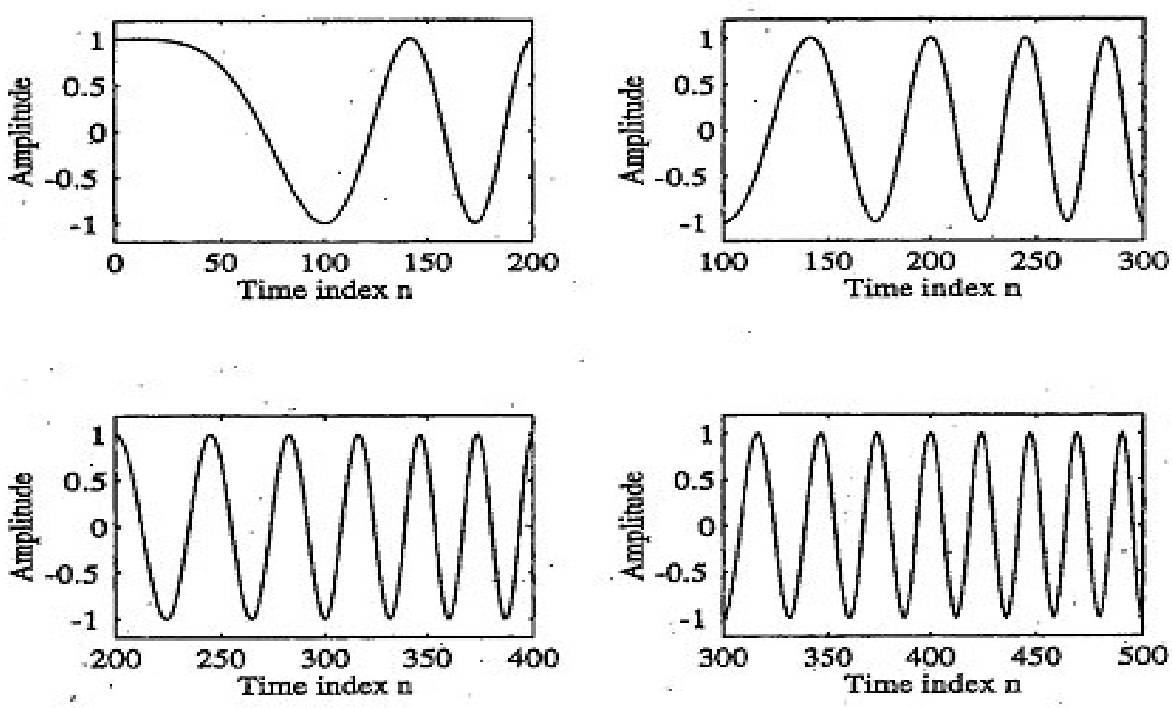
a) $f_1 = 0.28$ => k=
b) $f_1 = 0.29$ => k=
c) $f_1 = 0.30$ => k=
d) $f_1 = 0.31$ => k=

❖ **<u>Comments</u>:**

❖ Application of the DTFT to the Short-Time Fourier Transform



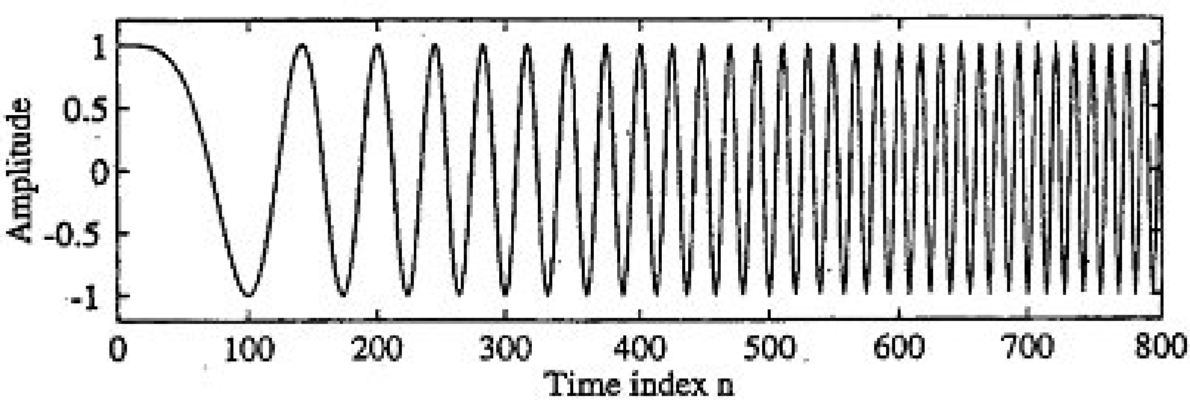
First 800 samples of a casual chirp signal $\cos(\omega_o n^2)$ with $\omega_o = 10\pi \times 10^{-5}$.



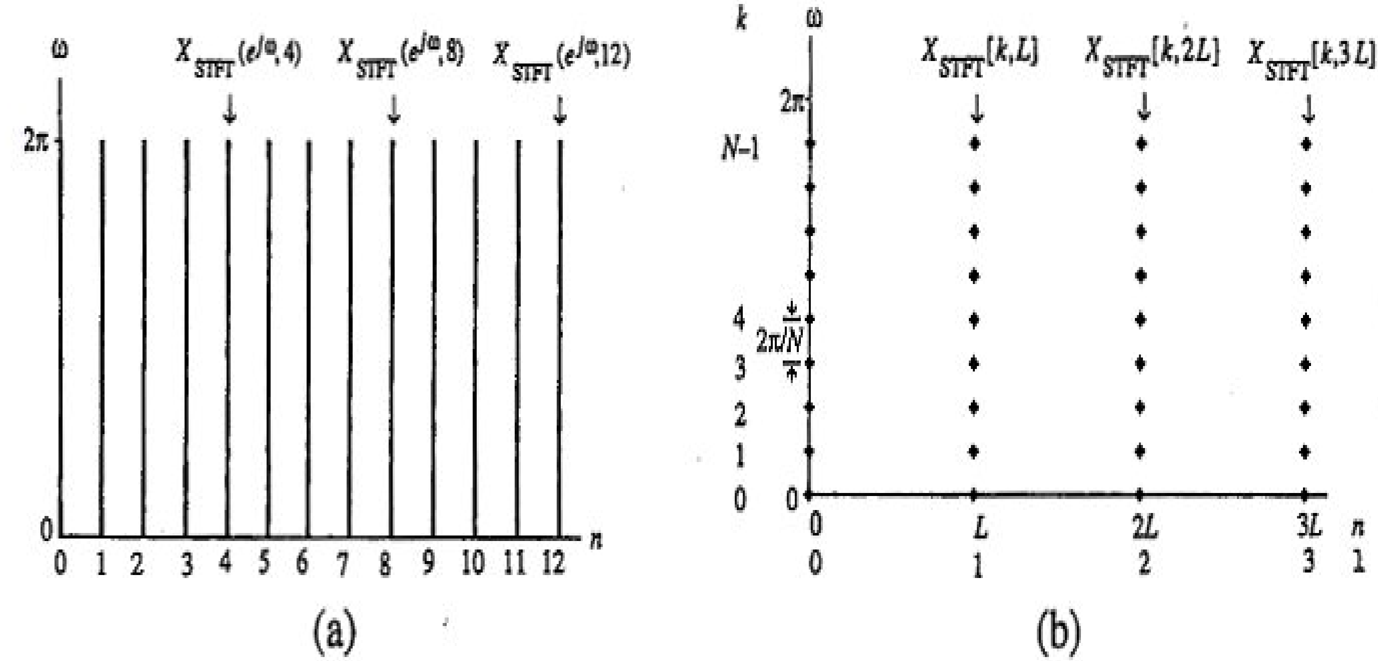
Various frames of length 200 samples extracted from the chirp signal

Note: Need to preserve the time-varying information of the frequency.

How to do that?



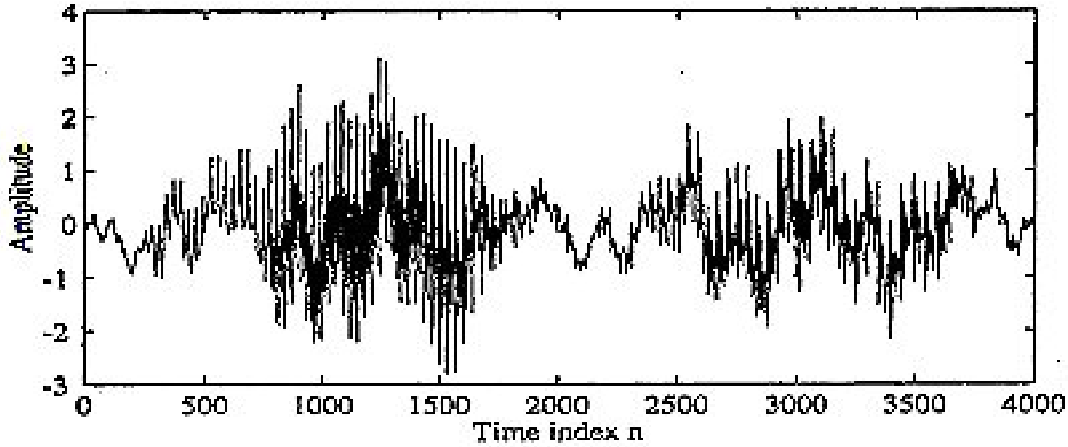
First 800 samples of a casual chirp signal $\cos(\omega_o n^2)$ with $\omega_o = 10\pi \times 10^{-5}$.



Sampling grid in the (ω,n)-plane for the sampled STFT XTFT[k,n], for N=9 & L=4

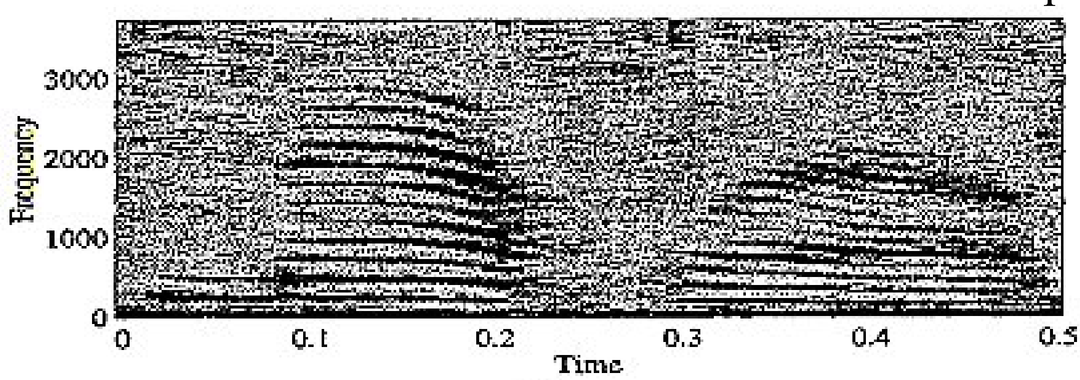
❖ Applications to Speech Processing

Time domain signal



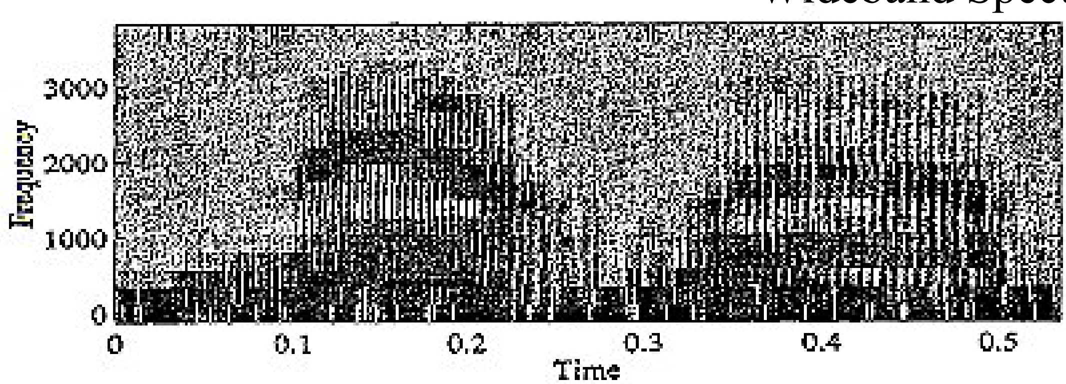
(a)

Narrowband Spectrogram



(b)

Wideband Spectrogram



(c)