III. Fourier Series

Response of LTI systems to complex exponentials
Fourier series
Fourier series property for real periodic x(t)
How to define the Fourier series coefficients
Fourier series coefficient properties
Truncation of Fourier series - Gibb’s phenomenon
Fourier series convergence - Dirichlet’s conditions
Fourier series for discrete time periodic signals
- Definitions
- Partial synthesis issues

Fourier series and LTI systems
Filtering
Application to amplitude modulation

III. Fourier Series

Basic Idea

- to be able to represent a signal as a linear combination of basic signals
- to take advantage of LTI system properties

1) Response of LTI systems to complex exponentials

$$x(t) = e^{st}$$

$$\Downarrow$$

$$y(t) =$$

$$x(t) = \sum a_i \, e^{s_i t}$$

$$\Downarrow$$

$$y(t) =$$

2) Fourier Series (for a periodic signal)

- Periodic signal $\Rightarrow$

- $x(t)=\sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t}$ periodic? ; period?

$$x(t)=\sum_{u=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \quad \text{; Fourier series representation}$$

$k=0 \quad \rightarrow$

$k=\pm 1 \quad \rightarrow$

$k=\pm 2 \quad \rightarrow$

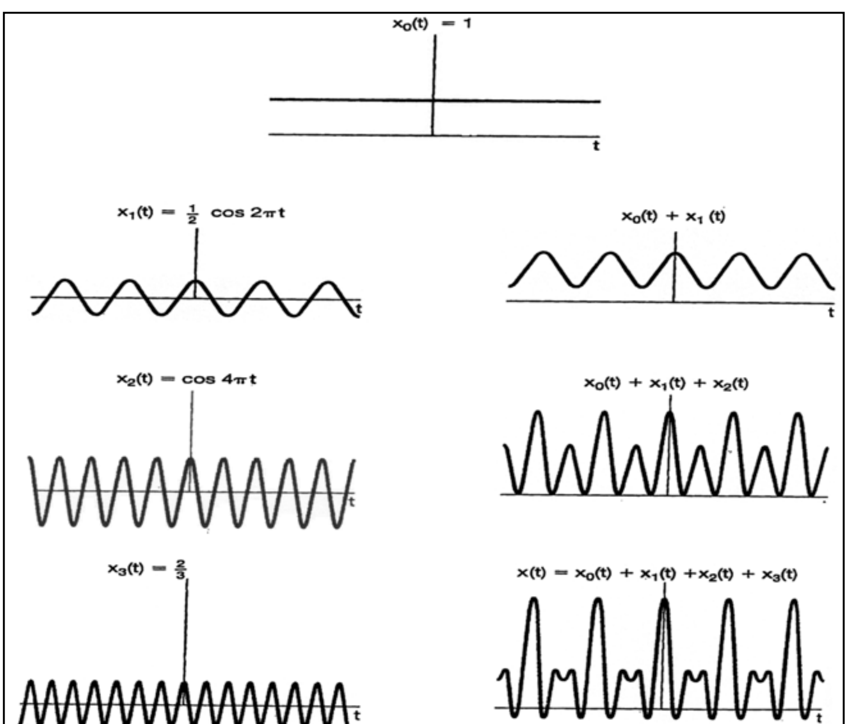


Figure 3.4 Construction of the signal $x(t)$ as a linear combination of harmonically related sinusoidal signals.

Example $x(t) = \sum_{k=-3}^{3} a_k \, e^{jk\omega_0 t}$

3) Fourier Series Property for Real, Periodic $x(t)$

$x(t) = x^*(t)$

4) How to Define a_k?

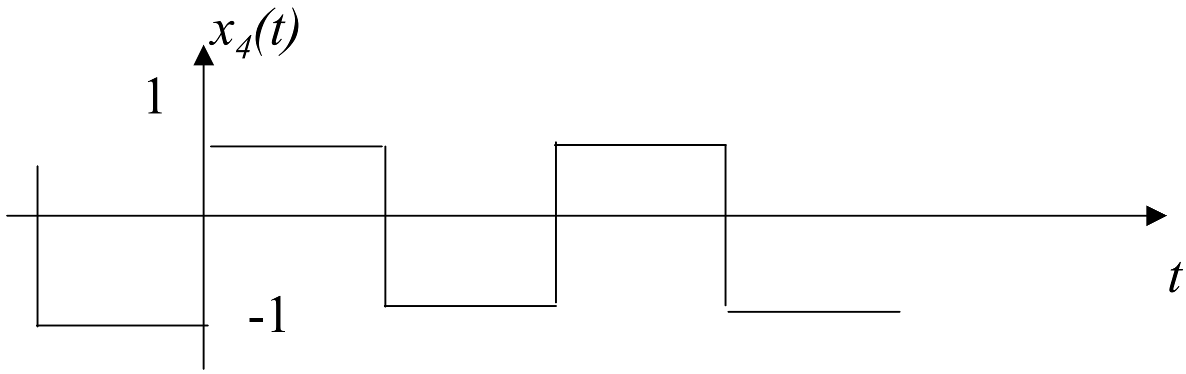
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{jk\omega_0 t}$$

- Examples: compute the Fourier series decomposition

$x_1(t)=\cos(2t)$

$x_2(t)=3\sin(3t+\pi/3)$

$x_3(t)=\cos(5t)-2\sin(15t)$



5) Fourier Series Coefficient Properties

$$x(t) \quad \text{real + even} \quad \Rightarrow \quad a_k \text{ real}$$

$$x(t) \quad \text{real + odd} \quad \Rightarrow \quad a_k \text{ imaginary}$$

- Proof:

$$x(t) \longleftrightarrow a_k$$
$$y(t) \longleftrightarrow b_k$$

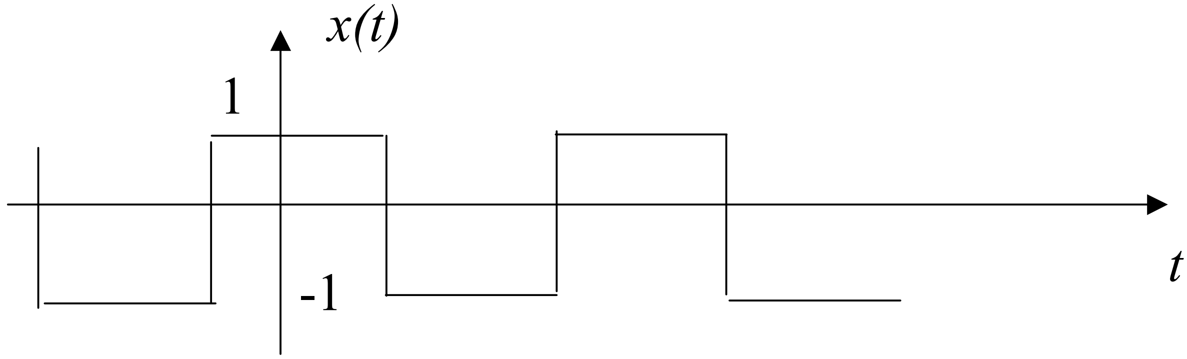
x(t) and y(t) with same period T

Property	**Signal**	**Coefficients**
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-shift	$x(t - t_0)$	$a_k\, e^{-jk\omega_0 t_0}$
Time reversal	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}

Parseval's relation: $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |a_k|^2$

Proofs:

- Example



6) Truncation of Fourier Series

– Gibbs' Phenomenon

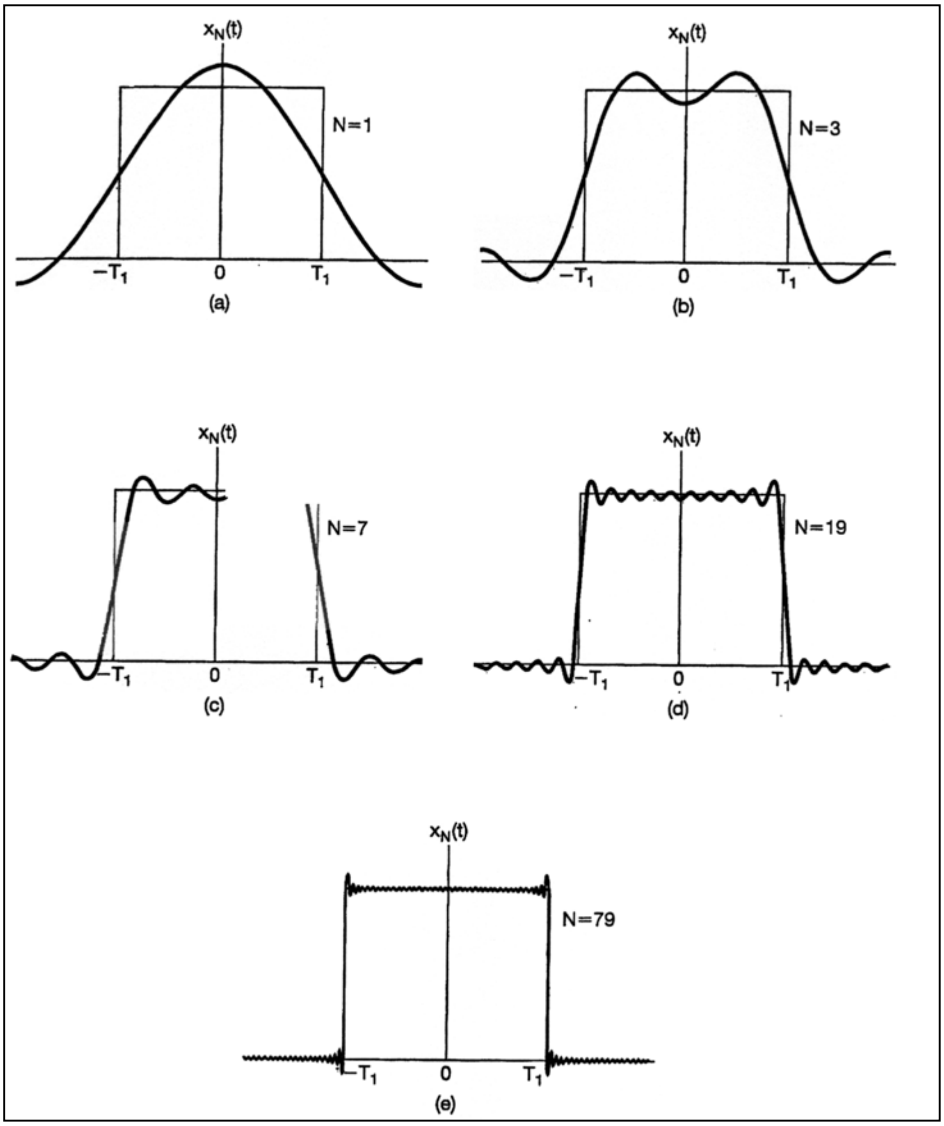


Figure 3.9 Convergence of the Fourier series representation as a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N\left(t\right) = \sum_{k=-N}^{N} a_k\, e^{jk\omega_0 t}$ for several values of *N*.

7) Convergence of Fourier Series

❖ Not all periodic functions can be represented by Fourier series

Why?

- → $a_n = \frac{1}{T} \int x(t) e^{-jn\omega_0 t} dt$ may DV
- → series may not CV to the original function

❖ Conditions for CV of series: <u>Dirichlet's Conditions</u>

(1) $x(t)$ is absolutely integrable over and period T_0
↳

(2) $x(t)$ has bounded variations
↳ no more than a finite number of maxima or minima during T

(3) $x(t)$ has a finite number of discontinuities

11) Fourier Series for Discrete-Time Periodic Signals

- <u>Differences</u> with results obtained for continuous-time periodic signals

- <u>Recall</u> – periodic discrete-time signal $x[n]$ defined as:

  – periodic continuous-time signal $x(n)$ expressed in Fourier series expansion as:

  $x(t) =$

  – periodic discrete-time complex exponential defined as:

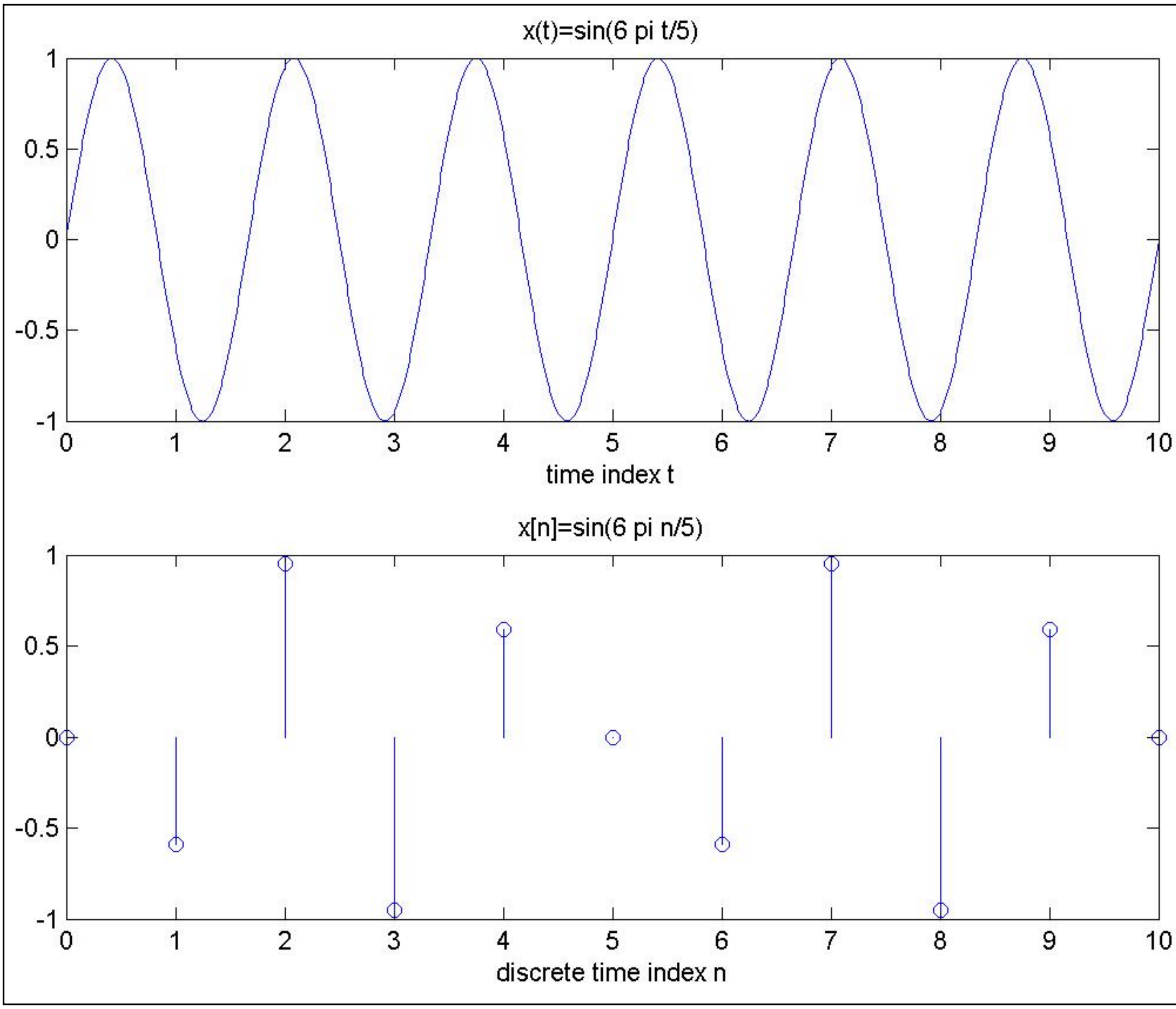
$$\boxed{\begin{array}{l} x[n] = \\ a_k = \end{array}}$$

- Examples:

$$x[n] = \sin\left(\frac{2\pi}{5}n\right)$$

$$y[n] = \sin\left(\frac{6\pi}{5}n\right)$$

$$z[n] = 1 + \sin(2\pi/5) + 3\cos(2\pi n/5) - \cos(3\pi n/5 + \pi/3)$$



x(t)=sin(6 pi t/5)
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10
time index t
x[n]=sin(6 pi n/5)
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10
discrete time index n

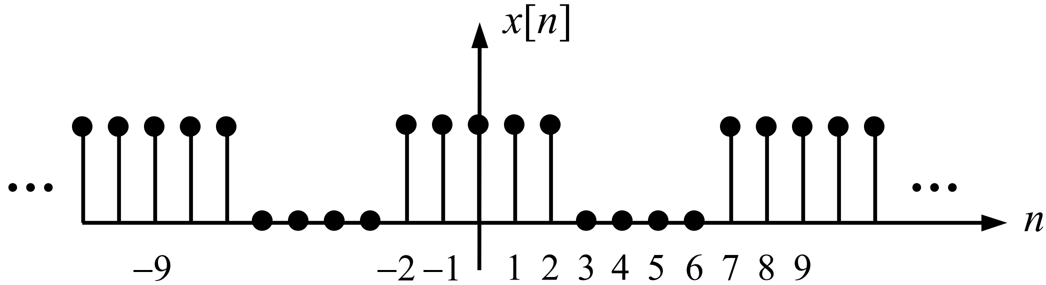


Figure 3.16 Discrete-time periodic square wave.

$$x[n] = 1 \; for \quad -N_1 \le n \le N_1 (N_1 = 2) + \text{periodic (N=9)}$$

$$\rightarrow a_k = \frac{2N_1+1}{N} \; ; \quad k = 0, \pm N, \pm 2N \cdots (3.105)$$

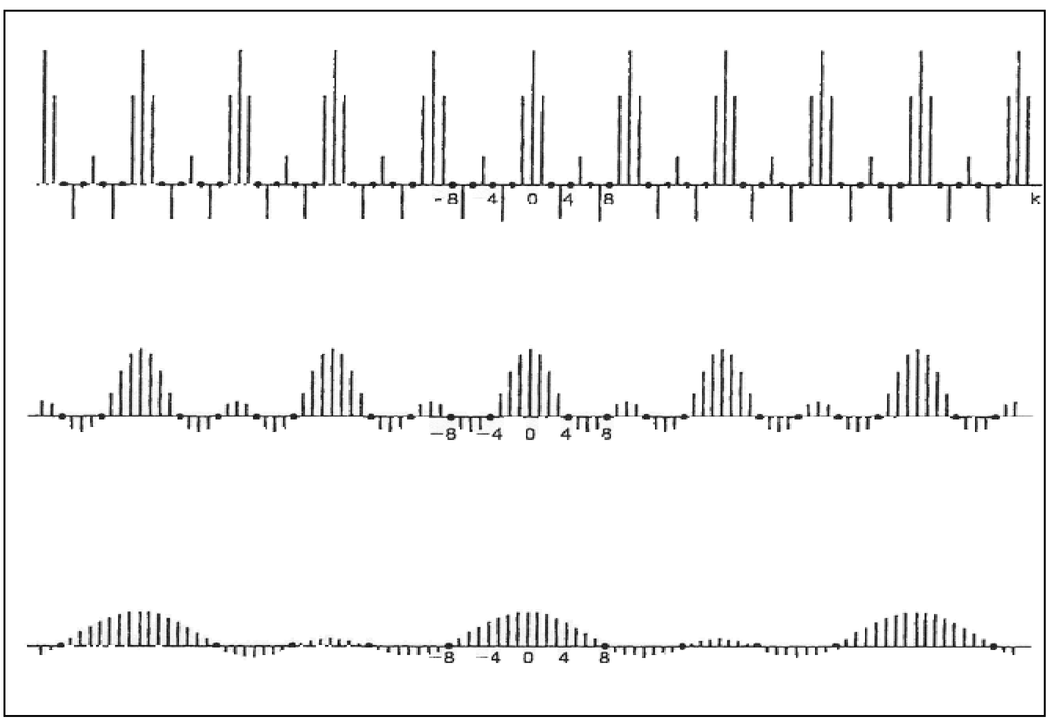


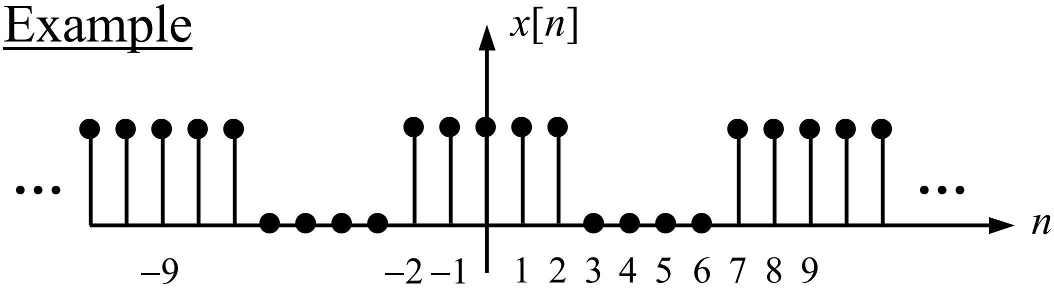
Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $\underbrace{2N_1+1=5}_{N_1-2}$ and $N = 10$; (b) $N = 20$; and (c) $N = 40$.

❖ Partial Synthesis Issues

$$x[n] = \sum_{\langle N \rangle} a_k \, e^{jk2\pi n/N} \qquad N\text{: period of } x[n]$$

$$= \sum$$

Example



M=1

M=2

M=3

M=4

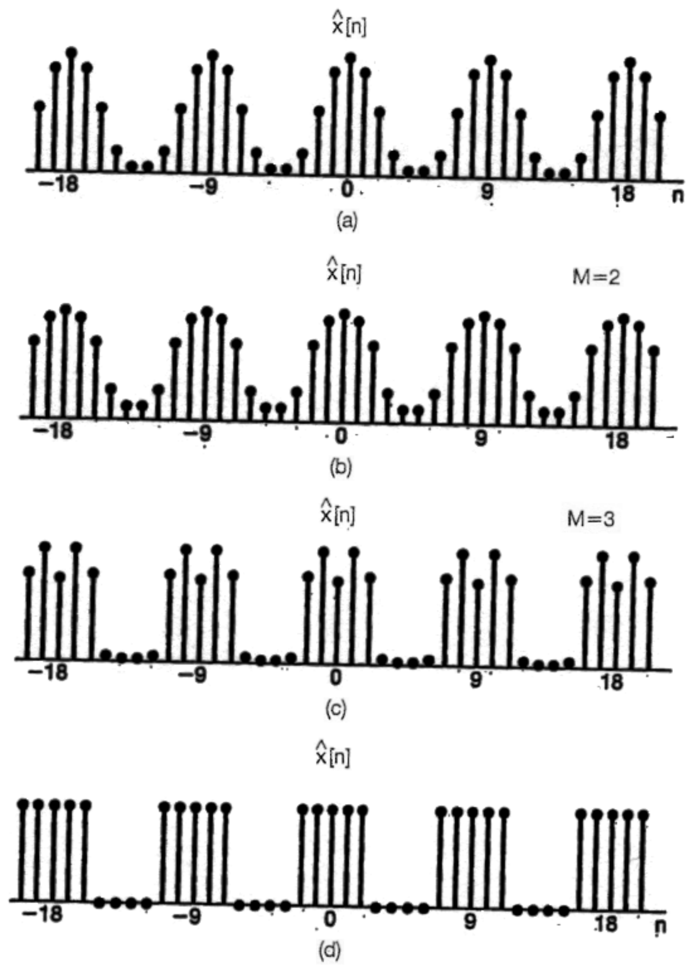


Table 3.1 Properties of Continuous-Time Fourier Series

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		Differentiation	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Differentiat		Integration	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Differentiation	Diffe	Different	I $a_k = a^*_{-k}$ $\Re e\{a_k\} = \Re e\{a_{-k}\}$ In $\Im m\{a_k\} = -\Im m\{a_{-k}\}$ $|a_k| = |a_{-k}|$ $\sphericalangle a_k = -\sphericalangle a_{-k}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Table 3.2 Properties of Discrete-Time Fourier Series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$

9) Fourier Series and LTI Systems

- Recall if $x(t) = e^{st}$ $\longrightarrow$ $\boxed{\text{LTI}}$ $\quad y(t) = e^{st} H(s)$

Called eigenfunction

$$H(s) = \int e^{-st} h(\tau)\, d\tau$$

$$x[n] = z^n \quad \rightarrow \quad y[n] = \sum x[n-k] \cdot h[k]$$

$$=$$

$$\Rightarrow \qquad H(z) =$$

- $\left.\begin{matrix} H(s) \\ H(z) \end{matrix}\right\} \longrightarrow$ called _______________

- for $s = j\omega$ $\qquad H(j\omega) =$

  called _________________

- if $x(t)$ expressed in a Fourier series decomposition

$x(t) =$

$\Downarrow$

$y(t) =$

$x[n] =$

$\Downarrow$

$y[n] =$

Example: • $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

• LTI system with $h(t) = e^{-t}\,u(t)$

(1) Compute the frequency response $H(j\omega)$

(2) Compute $y(t)$

Example: $x\left(n\right)=\cos\left(\frac{2\pi n}{10}\right)$, $h\left[n\right]=0.5^{n}u\left(n\right)$ LTI

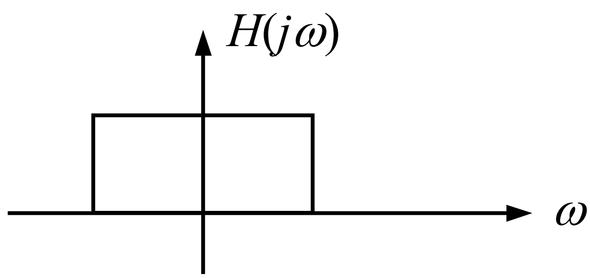
(1) Period of $x[n]$

(2) Frequency response $H(j\omega)$

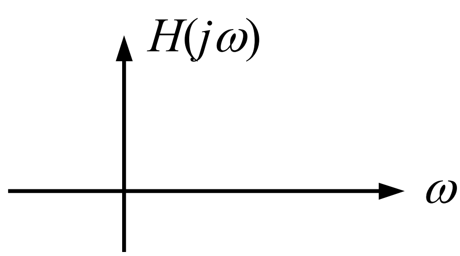
(3) Filter output $y[n]$ to $x[n]$

10) Filtering

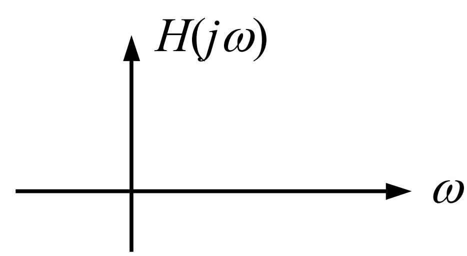
- Goal: to charge the relative amplitudes of signal frequency components
- Different types of filters



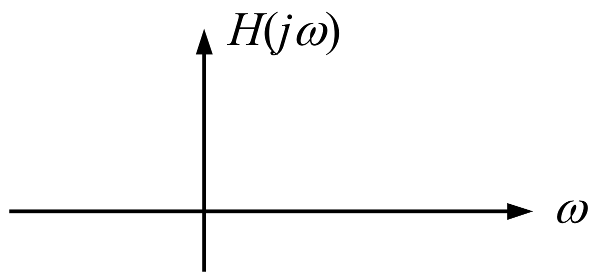
(1) Low Pass Filter (LP)



(2) Bandpass Filter (BP)

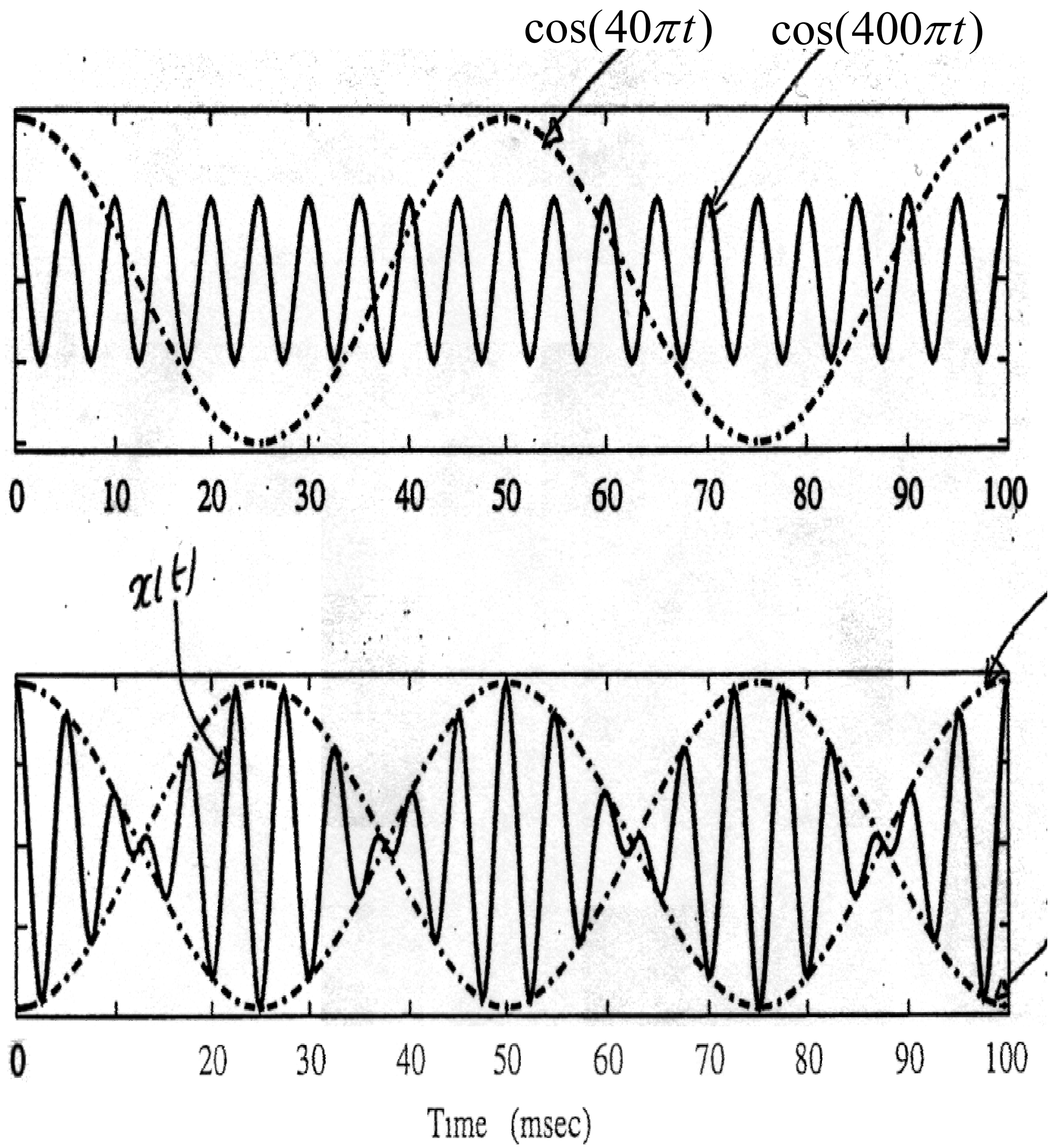


(3) High Pass Filter (HP)



10) Applications to Amplitude Modulation (AM)

$$x(t) = \cos(40\pi t).\cos(400\pi t)$$



Questions:

1) Is $x(t)$ periodic ? If so, what is the period ?

2) Compute the Fourier series decomposition of $x(t)$

3) Plot the spectrum of $x(t)$