Z-Transforms

❖ Recall:

— Convolution operation may be cumbersome

— Alternative: transform problem to a different domain where it is easier to compute

1) Definition

The z-transform of the sequence $x(n)$ is defined as:

❖ Example: Compute the z-transform of

1) $x(n) = a^n u(n)$

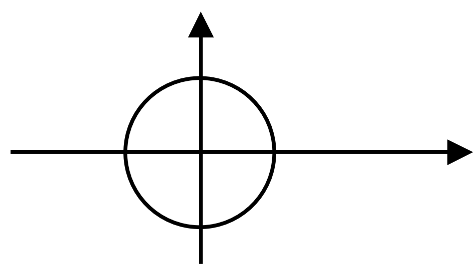
2) $x(n) = a^n u(-n-1)$

3) $x(n) = a^{|n|}$ $\quad |a|<1$

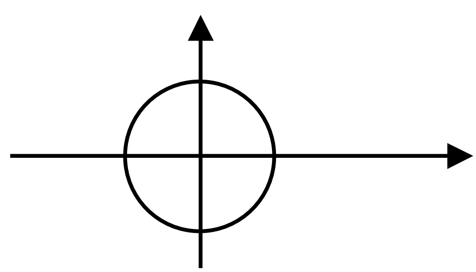
2) <u>Region of Convergence</u> (ROC): Determine the range of values z over which $X(z)$ is defined.

Causal/anti-causal sequence ROC

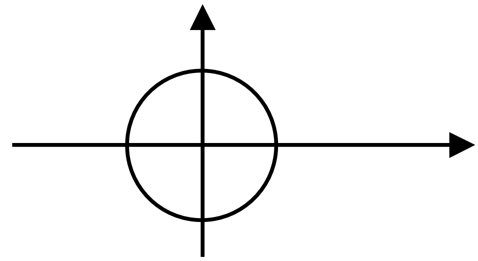
- $x(n) = a^n u(n)$



- $x(n) = a^n u(-n-1)$



- $x(n) = 0.3^{|n|}$



3) **Basic Transform Pairs:**

$$1)\, x(n) = A\delta(n)$$

$$2)\, x(n) = Au(n)$$

$$3)\, x(n) = Aa^n \qquad n \geq 0$$

$$4)\, x(n) = Aa^n e^{jn\theta}; \qquad n \geq 0$$

$$5)\, x(n) = Aa^n \cos(\theta n); \qquad n \geq 0$$

$$6)\, x(n) = \cos(\theta n)$$

$$7)\, x(n) = 2\,|C|\cos(n\theta + \langle C)$$

$$8)\, x(n) = 2\,|C|\,a^n \cos(n\theta + \langle C)$$

4) **Properties:**

1) Linearity property

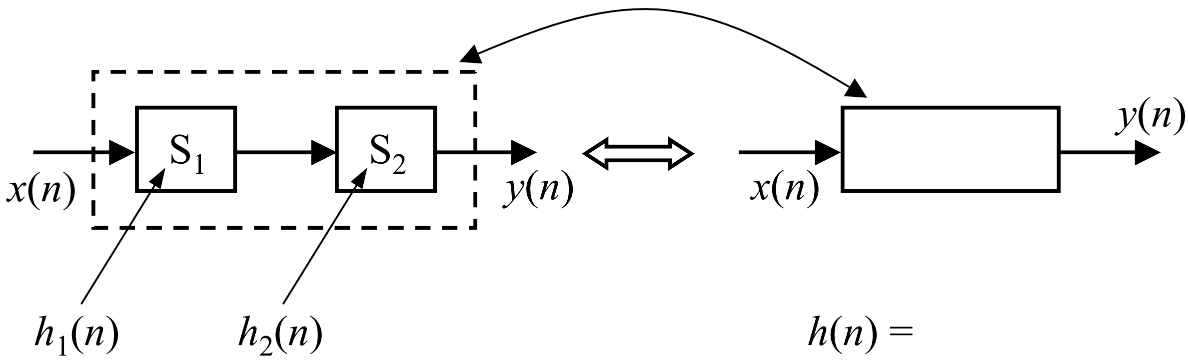
2) Shifting property

3) Multiplication by *n* and derivative property

4) Convolution property

5) Application of the Convolution Property

Recall



Example: $h_1(n) = h_2(n) = \delta(n) + \delta(n-4)$

Compute H(z), $h(n)$

Example: The impulse response to a LTI system is given by:

$$h(n) = \left(\frac{1}{2}\right)^n; \qquad 0 \le n \le 3$$

- With the input $x(n) = 2$, $0 < n < 5$

Compute the system output to $x(n)$

1) using the time-domain method
2) using the z-transform

6) Transfer Function

- Definition: the system transfer function is defined as

$$T(z) = \frac{\text{z-transform of output sequence}}{\text{z-transform of input sequence}}$$

$$=$$

- Two possible options to compute the transfer function:

(1) Transfer function from definition

Example: $x(n) = 2; \quad 0 < n < 5$

$$y(n) = 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$$

Compute the system transfer function

(2) Transfer function from unit impulse response

$$x(n) = \delta(n) \;\rightarrow\; y(n)$$

$$z\left[\delta(n)\right] = \qquad\qquad \Rightarrow \; z\left[y(n)\right] =$$

Notation change: the system transfer function is noted as:

Example:

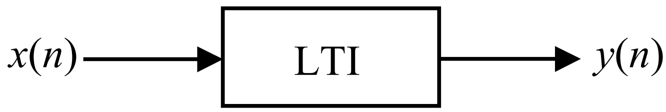
$$1)\, h(n) = 2(0.3)^n u(n)$$

$$2)\, h(n) = \delta(n) + 2\delta(n-2)$$

$$3)\, h(n) = (0.6)^n \cos\left(n\frac{\pi}{3} - 0.2\pi\right) u(n)$$

Compute the system transfer function

(3) Transfer function from the difference equation



Recall: The system I/O relationship is given by:

$$y(n) - \sum_{k=1}^{N} a_k \, y(n-k) = \sum_{\ell=0}^{L} b_l \, x(n-\ell)$$

$\Rightarrow$

Example:

$$y(n) - 2y(n-1) = 2x(n) - 0.5x(n-1)$$

Compute the transfer function

❖ Transfer Function and System Stability

- Recall a LTI system is used to be stable if

(1) in terms of $h(k)$:

(2) in terms of the initial condition (IC) response:

- How to relate the stability to the locations of the characteristic roots.

7) Graphical Design of Filters/Poles and Zeroes

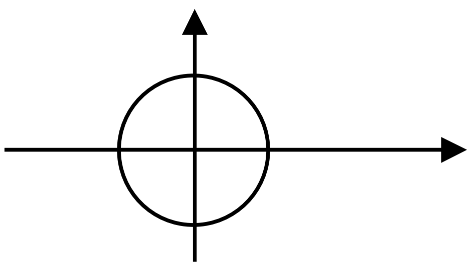
$$H(z) =$$

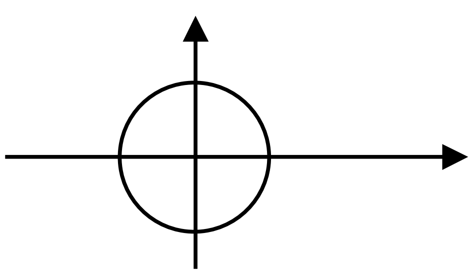
$$H\left(e^{j\theta}\right) = \frac{B\left(e^{j\theta}\right)}{A\left(e^{j\theta}\right)} = \frac{\sum_{i=0}^{L-1} b_i \, e^{-j\theta i}}{\sum_{i=0}^{N-1} a_i \, e^{j\theta i}}$$

$$=$$

Example: $H_1(z) = 1 - 0.5z^{-1}$

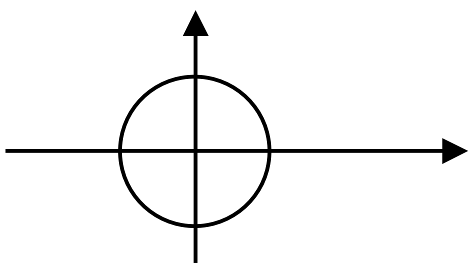
$$H_2(z) = \frac{1}{1 - 0.5z^{-1}}$$





Example:

$$H_1(z) = \frac{z^2 - 1}{z^3 + 0.5z}$$



8) Inverse z-Transform

- Recall

$$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n) =$$

$$Z\left[x(n)\right] = X(z) \rightsquigarrow \quad Y(z) =$$

- Two main procedures available to compute the inverse z-transform:

(1) direct inversion formula (not done here)

(2) by tables and partial function expansion

(3) by long division

(3) by long division: useful only if we need a few samples, too complicated otherwise

Example:

$$H(z) = \frac{0.5z^2 + 0.5z}{z^2 - z + 0.5}; \qquad |z| > 0.707$$

Example:

$$H(z)=\frac{0.5z^2+0.5z}{z^2-z+0.5}; \qquad |z|>0.707$$

(2) by tables and partial function expansion:

$$X(z)=\frac{N(z)}{D(z)}=k\,\frac{(z-z_1)\cdots(z-z_L)}{(z-p_1)\cdots(z-z_p)}$$

- Recall:

$$x(n)=a^n u(n) \Leftrightarrow X(z)=\frac{z}{z-1}$$

- Goal: to expand X(z) in terms with known transform pairs so that the inverse transform can be easy to obtain

- Three cases:

  a) Single poles L<P
  b) Single poles L>P
  c) Multiple poles

<u>(a) Assume simple poles and $L < P$</u>

Step 1: Define G(z)=H(z)/z

Step 2: Compute the partial fraction expansion of G(z)

$$G\left(z\right) = C_1 \frac{1}{z - p_1} + C_2 \frac{1}{z - p_2} + \cdots + C_p \frac{1}{z - p_p}$$

Step 3: Compute the C_i constants as:

$$C_i = G(z)(z - p_i)\Big|_{z=p_i}$$

Step 4: Compute H(z)

$$H\left(z\right) = zG(z) = C_1 \frac{z}{z - p_1} + C_2 \frac{z}{z - p_2} + \cdots + C_p \frac{z}{z - p_p}$$

Step 5: Back-transform to the time domain

$$h(n) = C_1 p_1^n u(n) + \cdots C_p p_p^n u(n)$$

Example: $$H\left(z\right)=\frac{4z^{2}}{z^{2}-0.25}$$

$$H\left(z\right)=\frac{z}{z^{2}+1}$$

(b) Assume simple poles and $L \geq P$

Example:

$$H(z) = \frac{z}{z-0.5}$$

$$H(z) = \frac{z^2}{z-0.5}$$

(c) Assume repeated poles and $L < P$

Example:

$$H(z) = \frac{z}{(z-0.5)(z-1)^2}$$

- As before, define G(z)=H(z)/z

$$G(z) = H(z)/z = \frac{1}{(z-0.5)(z-1)^2}$$

$$= C_1 \frac{1}{z-0.5} + \underbrace{\frac{C_2}{z-1} + \frac{C_3}{(z-1)^2}}_{\substack{\text{2 terms for a pole} \\ \text{of multiplicity 2}}}$$

$\downarrow$

n terms for a pole of multiplicity *n*

$$G(z) = H(z)/z = \frac{1}{(z-0.5)(z-1)^2}$$

$$= C_1 \frac{1}{z-0.5} + \underbrace{\frac{C_2}{z-1} + \frac{C_3}{(z-1)^2}}$$

2 terms for a pole
of multiplicity 2

n terms for a pole of
multiplicity *n*

– for multiple poles p_i:

$$C_\ell = \frac{1}{(\ell-1)!}\left(\frac{d^{\ell-1}}{dz^{\ell-1}}\right)\left[G(z)(z-p_i)^{\ell}\right]\Big|_{z=p_i} \qquad \ell = 1, \cdots, L$$

Example: $$H(z) = \frac{z}{(z-0.5)(z-1)^2}$$

9) Solution of Difference Equations Using the z-Transform

- Recall

$$y(n) - \sum_{k=1}^{N} a_k \, y(n-k) = \sum_{\ell=0}^{L} b_\ell \, x(n-\ell)$$

$$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n)$$

- Restriction to causal systems $(n \geq 0)$

  We need to be careful about the impact of initial conditions

  Recall:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[x(n-m)] = z^{-m} X(z)$$

If $\quad x(n) \neq 0, n \geq 0$

$$Z[x(n-m)] =$$

Caution:

Initial conditions show up in the Z-transform of the difference equation

$y(n) - 1.01\, y(n-1) = x(n);$ $\qquad y(-1) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad x(n) = 10u(n)$

- Restriction to causal systems ($n \geq 0$)

Example:

a) $y(n) - 1.01\, y(n-1) = x(n);$ $\quad y(-1) = 0$, $\quad x(n) = 10u(n)$

b) $y(n) + y(n-2) = x(n) + x(n-1)$ $\quad y(-2) = -10$, $\quad y(-1) = 0$, $\quad x(n) = 10u(n)$

Basic z-Transform Pairs

Sequence	Transform	ROC
$\delta(n)$	1	all z
$u(n)$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\dfrac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta(n-m)$	z^{-m}	all except $0(m>0)$ $\infty(m<0)$
$a^n u(n)$	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u(-n-1)$	$\dfrac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$2\lvert C\rvert \cos(n\theta + \angle C)u(n)$	$\dfrac{Cz}{z-e^{j\theta}} + \dfrac{C^* z}{z-e^{-j\theta}}$	$\lvert z\rvert > 1$
$2\lvert C\rvert a^n \cos(n\theta + \angle C)u(n)$	$\dfrac{Cz}{z-ae^{j\theta}} + \dfrac{C^* z}{z-ae^{-j\theta}}$	$\lvert z\rvert > \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & ow \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$Cna^n u(n)$	$\dfrac{Cz}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-Cna^n u(-n-1)$	$\dfrac{Cz}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$