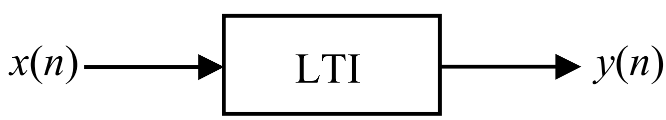
# III. Frequency Response & Filters

- <u>Motivation</u>
  - We have seen that the response of an LTI system to a combination of inputs is the linear combination of outputs.
  - Look at system outputs to a sinusoidal sequence.

## 1) Sinusoidal Steady State Response of LTI Systems




— $x(n) = e^{j\theta n}$ $\qquad\qquad$ $y(n) =$

$$=$$

— Definition $\quad$ The eigenvalue of the LTI system is defined as:

—The system eigenfunction is defined as:

- Steady-state response:

(1) $x(n)=e^{j\theta n}$ ==> $y(n)=$

(2) $x(n)=A\cos(\theta_0 n)$
==> $y(n)=$

(3) $x(n)= x(n)=A\cos(\theta_0 n+\alpha)$
==> $y(n)=$

- <u>Example</u>:
  $y(n)=x(n)+2x(n-1)$

Compute the output $y(n)$ to
  (1) $x(n)=\exp(j\pi n/3)$
  (2) $x(n)=\cos(2\pi n/7+\pi/5)$

- Steady state response plots

- Recall (Euler's Identities)

- Continuous time:

$$A\cos(\omega_0 t+\alpha)=$$

- Discrete time:

$$A\cos(\omega_0 n T_s+\alpha)=$$
$$=$$

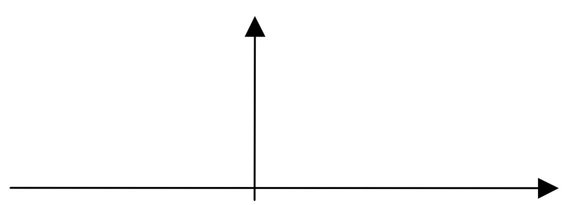
- Note the following facts:

1) For each sinusoid you need two complex frequencies: one positive and one negative

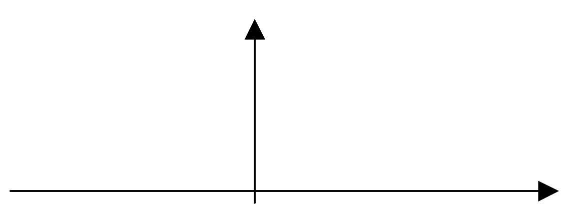
2) To each frequency we associate a magnitude and a phase term

$$Ae^{j\theta_0 n+\alpha}$$

magnitude

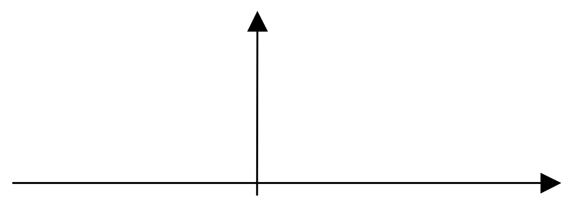


phase

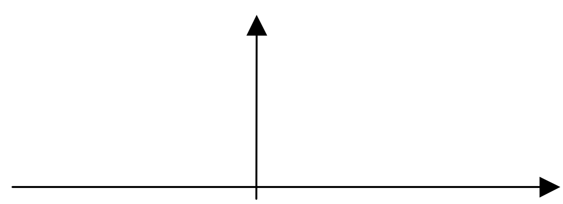


$$Cos(\theta_0 n+\alpha)=$$

magnitude



phase



Example:

$$x(n) = 2\cos\left(\frac{3\pi}{10}n + \frac{\pi}{3}\right) - 10\sin\left(\frac{8\pi}{5}n\right)$$

## 2) Properties of $H(e^{j\theta})$

$$H(e^{j\theta}) =$$

— $H(e^{j\theta})$ is periodic

— $H(e^{j\theta})$ has symmetry properties

(1) $\left|H(e^{j\theta})\right|$ is even

(2) phase of $H(e^{j\theta})$ is odd

(3) $H(e^{j\theta})$ is symmetric around $\pi$

Proof

Example: $$H\left(e^{j\theta}\right) = 2 + e^{-j\theta}$$

Plot $H(e^{j\theta})$

## 3) Frequency Range

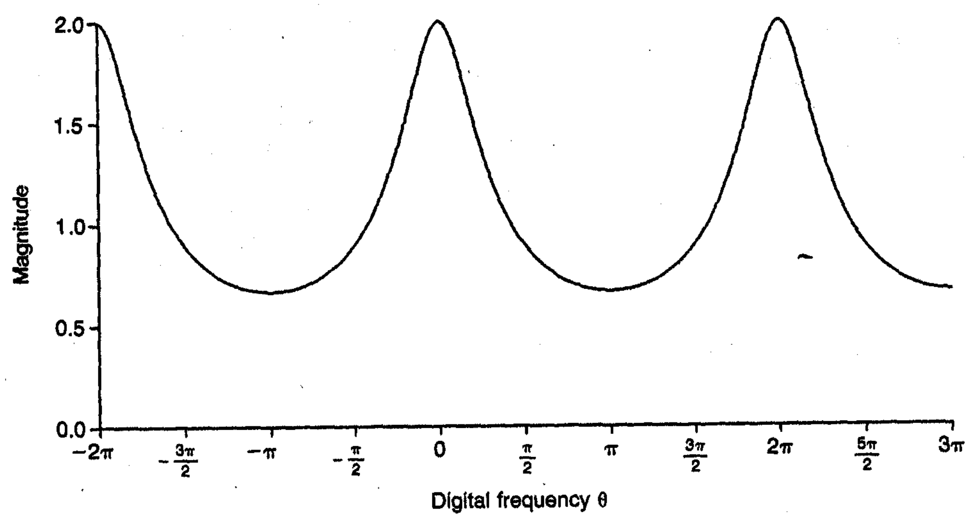
- Recall how digital and analog frequencies are related

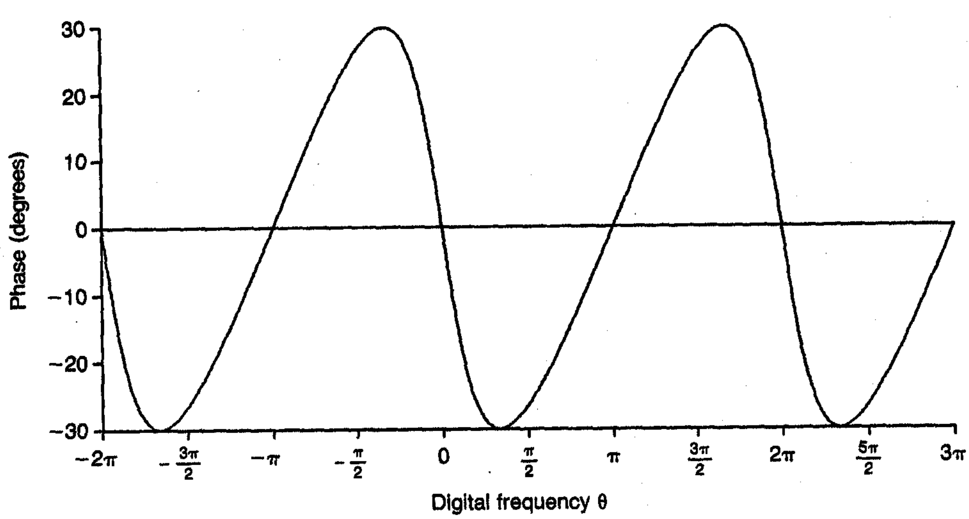
$$x(t) = \cos(\omega_0 t)$$
$$x(n) = x(nT_s) =$$

- Recall which range of digital frequencies is unambiguous,

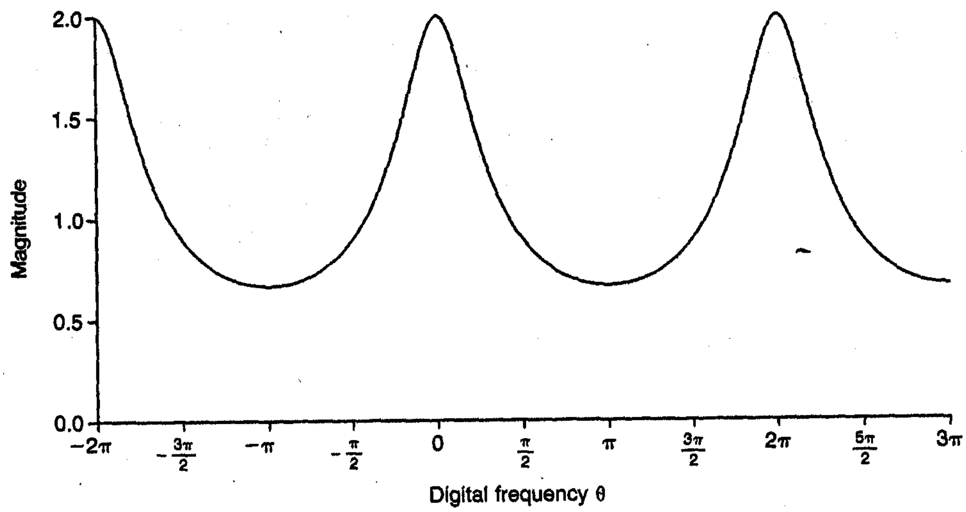
# Example:

$$H\left(e^{j\theta}\right)=\frac{2}{1-0.5e^{-j\theta}}$$



- Relationships between digital and analog frequencies



# 4) Definition of Decibels (dB)

– In engineering application the dB scale is usually used

Definition: dB are defined as:

| | | | | | |
|---|---|---|---|---|---|
| $M$ | 0.01 | 0.1 | 1.0 | 10 | 100 |
| $M_{dB}$ | -40 | -20 | 0 | 20 | 40 |

– Advantage of using a dB scale over a linear scale is:

# 5) **Plotting the frequency response using MATLAB**

- Given the frequency response:

$$H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = \frac{b_1 + b_2 e^{-j\omega} + \ldots + b_{m+1} e^{-jm\omega}}{a_1 + a_2 e^{-j\omega} + \ldots + a_n e^{-jn\omega}}$$

- Use the MATLAB function *freqz.m*

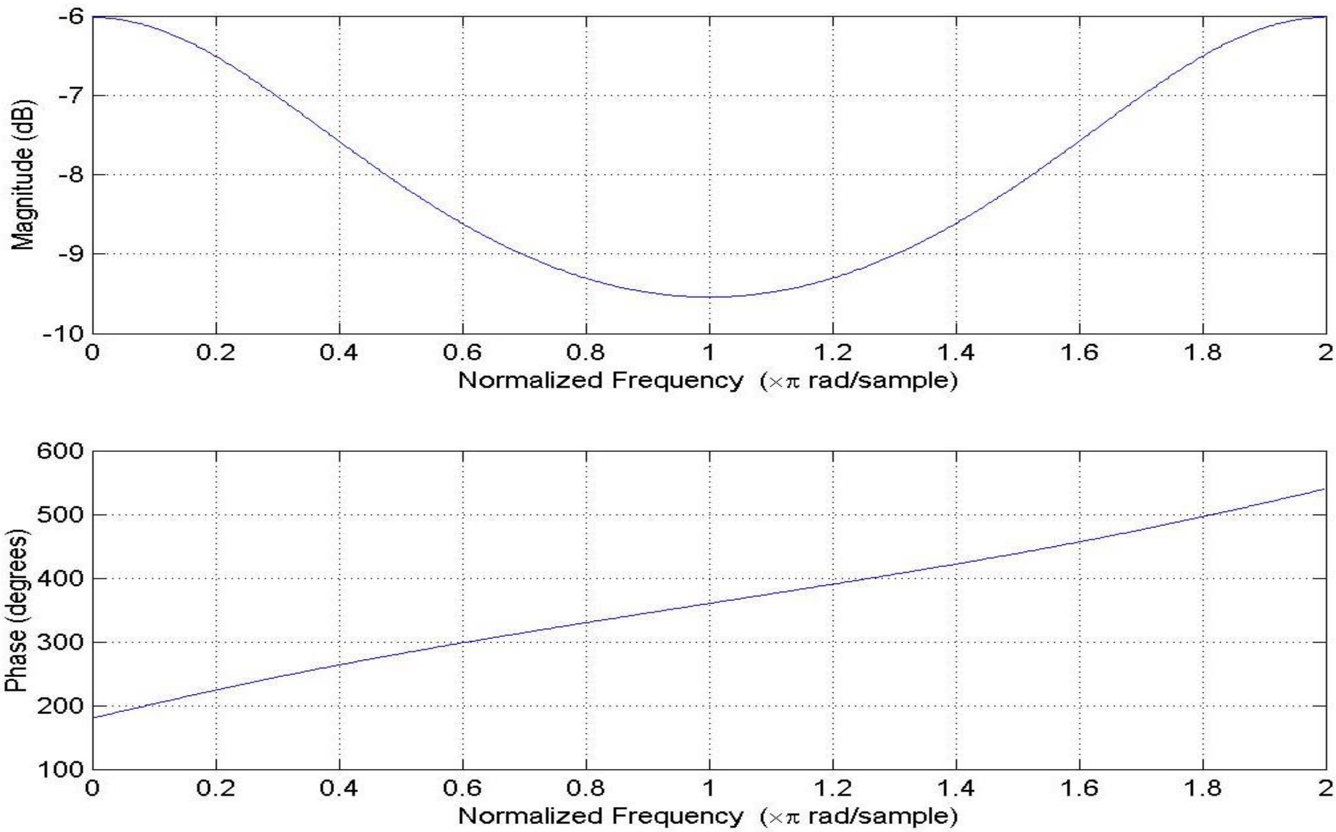
(1) *[H,W]=freqz(B,A,P)*
where P is the number of points plotted in [0,π]

(2) *[H,W]=freqz(B,A,P,'whole')*
where P is the number of points plotted in [0,2π]

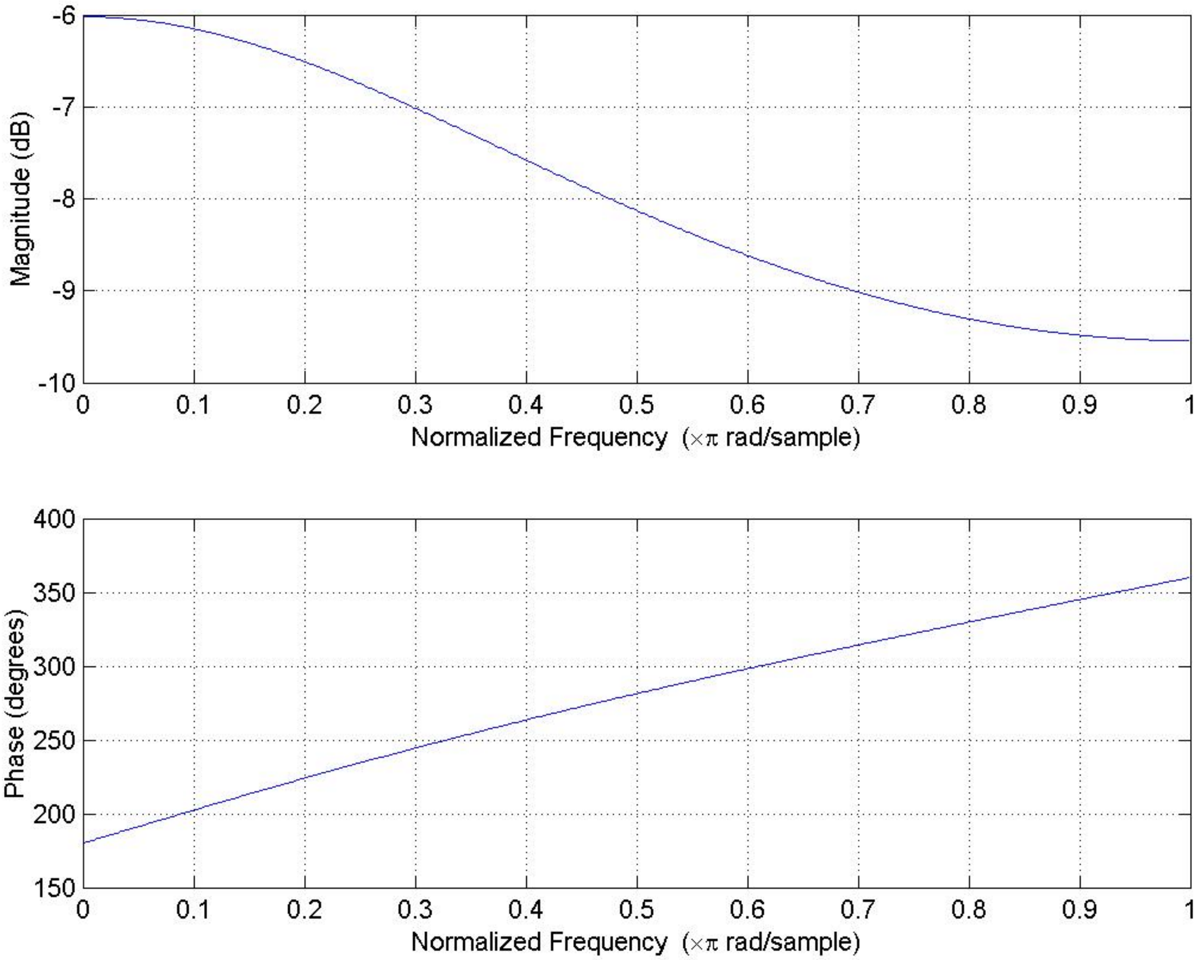
(3) *freqz(.....)* without output arguments does the plots

*freqz.m* applied to $H\left(e^{j\theta}\right)=\dfrac{2}{1-0.5e^{-j\theta}}$

*freqz(B,A,512,'whole'*)

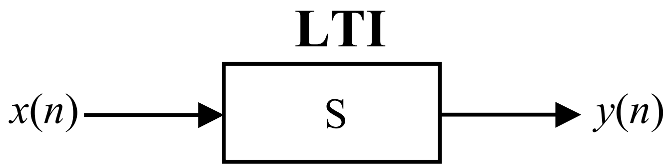


*freqz(B,A,512)*



-6
-7
-8
-9
-10
Magnitude (dB)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
400
350
300
250
200
150
Phase (degrees)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)

## 5) Frequency Response Function Defined from the Difference Equations




$$y(n) - a_1 y(n-1) + b_L x(n-L) - \cdots - a_N y(n-N) = b_0 x(n) + \cdots$$
$$+ b_L x(n-L)$$

— Recall when:

$$x(n) = e^{j\theta n} \rightarrow y(n) = x(n) * h(n)$$
$$= \sum h(m) x(n-k)$$
$$=$$
$$=$$

— Note when:

$$x(n) = e^{j\theta n} \rightarrow x(n-k) =$$
$$y(n) = \qquad \rightarrow y(n-k) =$$
$$=$$
$$=$$

Example: You are given the IO relationship

1) $$y(n)-0.5y(n-1)+0.3y(n-2) = x(n)-0.2x(n-1)$$

2) $$y(n)=\frac{1}{L+1}\sum_{k=0}^{L}x(n-k); \qquad (L=4)$$

- Compute the system frequency response
- Check whether the system is stable
- Plot the magnitude and phase of $H(e^{j\theta})$
- Find $y(n)$ when $x(n) = 0.5 \cos(\pi n/4)$
- Plot the system block diagram

Example: Assume the LTI system has the following impulse response

$$h(n) = \delta(n) + \delta(n-1)$$

1) Compute the system frequency response. Is the system stable?

2) Plot the system frequency response.

3) Compute the output to $X(n) = e^{j(\pi/2)n}$

4) Compute the output to

$$X(n) = 10\cos(\pi n/10) + 10$$

Example: Assume the LTI system has the following unit impulse response

$$h(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad 0 \leq a < 1$$

1) Is the system stable?

2) Compute the system frequency response.

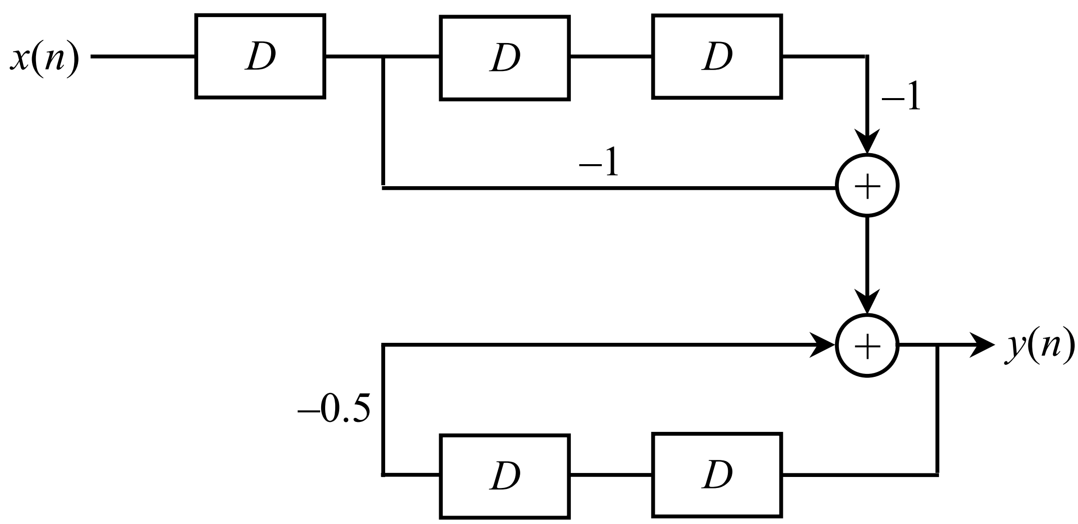
3) Plot the system frequency response.

4) Compute the output to the input $x(n) = 2$

5) Compute the output to the input $x(n) = \cos(\theta_0 n)$

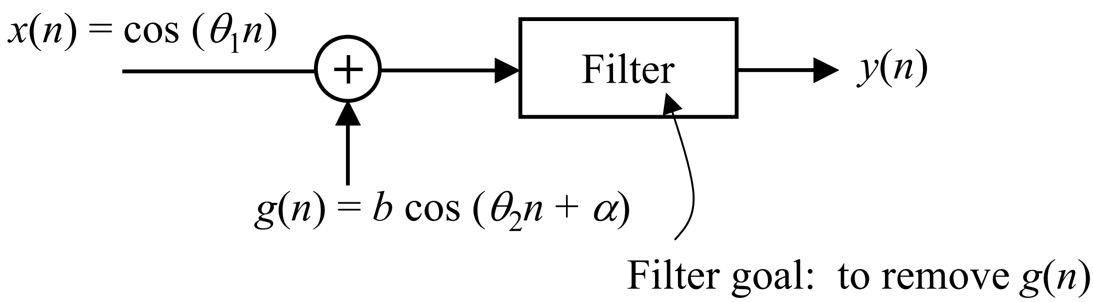
Example: You are given the following block diagram

- Find the frequency response
- Plot the frequency response



# 6) <u>Filters Definition</u>

<u>Definition</u>: A digital filter is a system which transforms a sampled signal into a second signal to remove an unwanted contribution.




assume $$x(n) = a\cos(\theta_1 n) = 10\cos\left(\frac{\pi n}{20}\right)$$

$$g(n) = b\cos(\theta_2 n + \alpha) = 4\cos\left(\frac{\pi n}{2} + \frac{\pi}{6}\right)$$

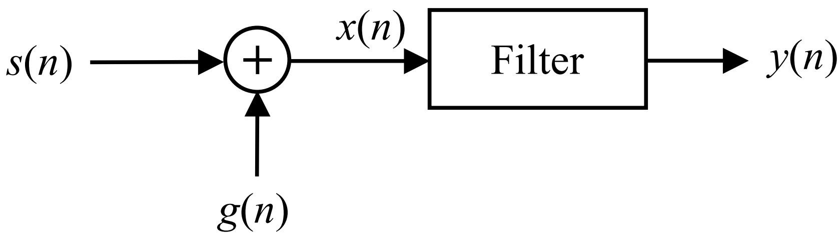
assume $$H\left(e^{j\theta}\right) = 1 + 0.9e^{-j2\theta}$$

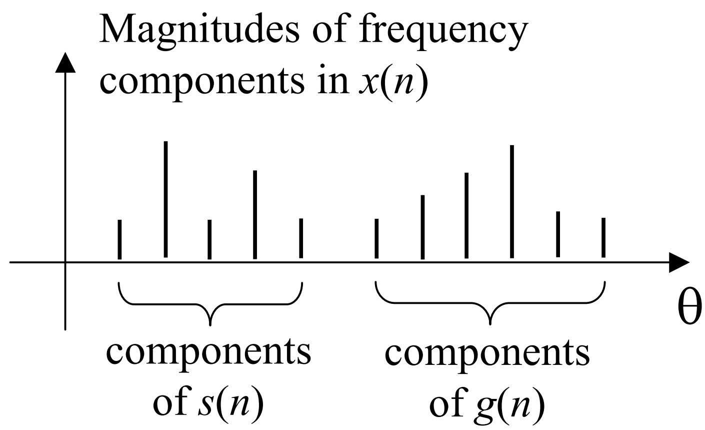
1) Plot magnitude and phase of frequency response.

2) Compute the output of the filter.

# 7) <u>Ideal Filters</u>

- Assume




- Goal of filter: pass components due to $s(n)$;
  kill components due to $g(n)$

Ideally we want $y(n) = s(n-k)$

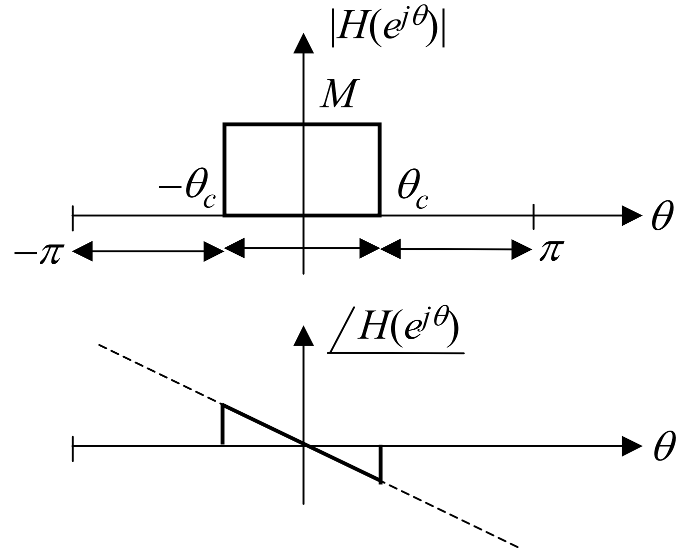
where

$$s(n) = \sum_{m=1}^{M_1} S_m \cos(\theta_m n);$$

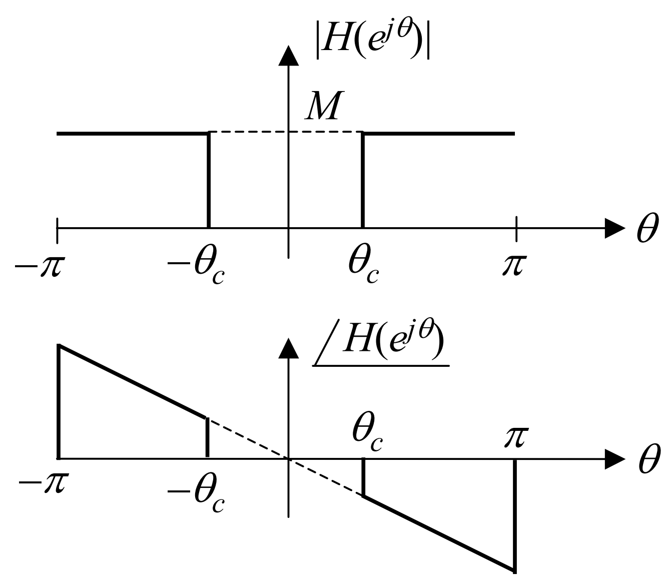
$$g(n) = \sum_{m=M_1+1}^{M_2} G_m \cos(\theta_m n)$$

# Filter Characteristics

**Lowpass**

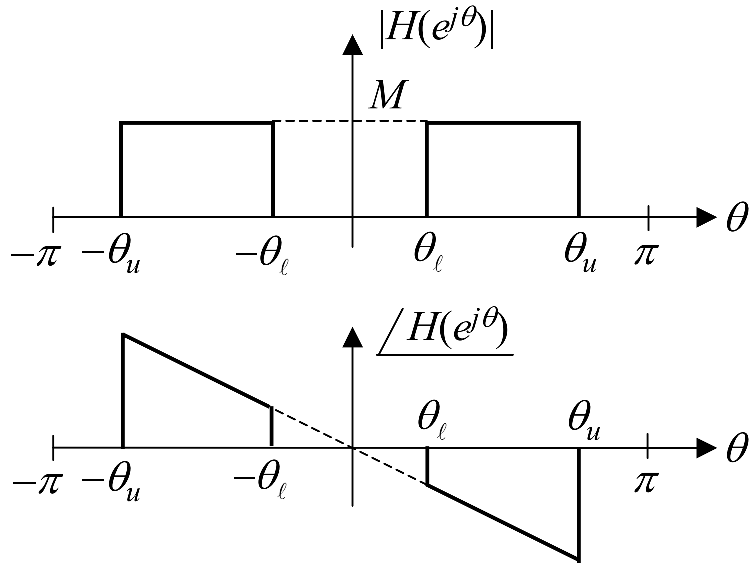



**Highpass**

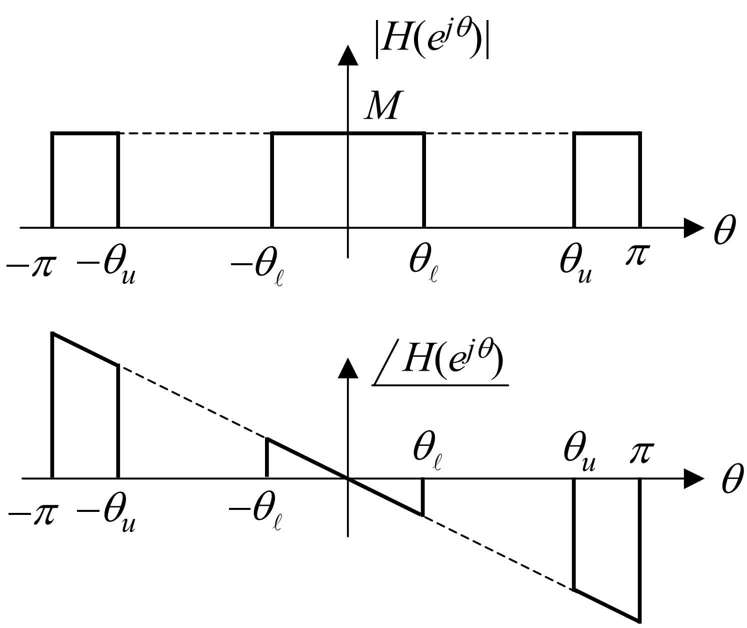


# Filter Characteristics (cont')

**Bandpass**




**Bandstop**



Example: You are given an analog signal with maximum frequency 50KHz. Design the digital filter which passes all frequencies in the range 10 to 20 KHz.

Example: You are given the input sequence $x(n)$ and output sequence $y(n)$ to a filter

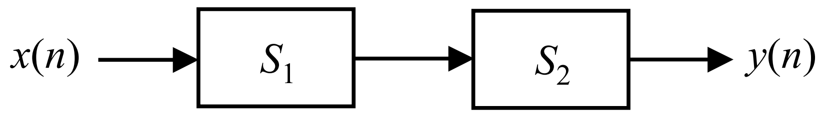
$$x(n) = 35\cos\left(\frac{\pi n}{5}\right) + 20\cos\left(\frac{\pi n}{2} - \frac{\pi}{4}\right) - 3\cos\left(\frac{2\pi n}{3} + \frac{\pi}{6}\right)$$

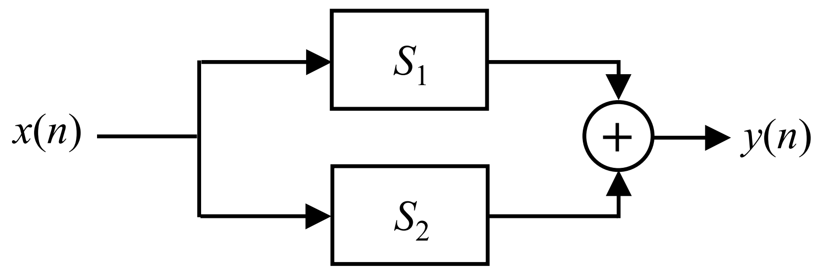
$$y(n) = 10.5\cos\left(\frac{\pi n}{2} - \frac{\pi}{2}\right) + 3\cos\left(\frac{2\pi n}{3}\right)$$

Plot the frequency response characteristics of the filter needed for this IO relationship.

# 8) <u>Frequency Response of Interconnected Systems</u>

Recall



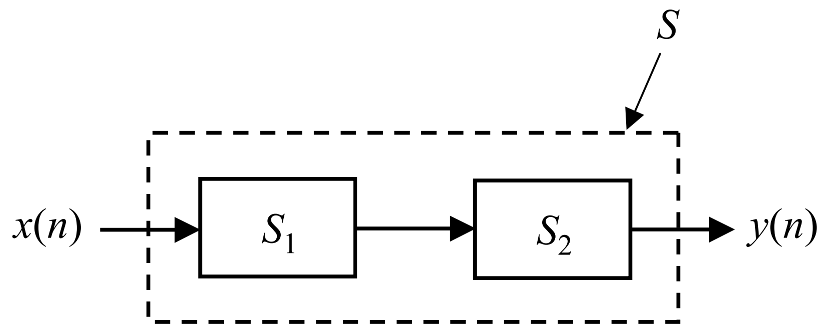

Assume

$$H_1\left(e^{j\theta}\right) = M_1 e^{jP_1}$$

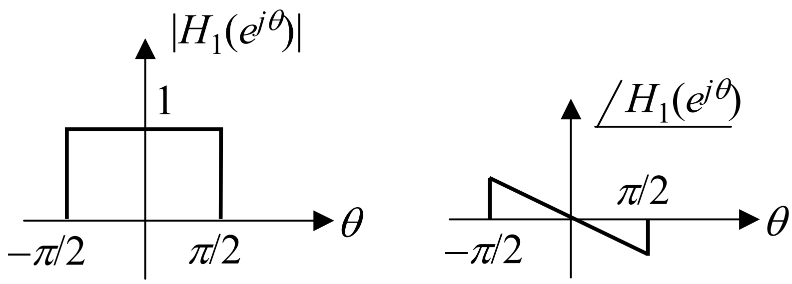
$$H_2\left(e^{j\theta}\right) = M_2 e^{jP_2}$$

$$x(n) = A\cos(\theta n)$$

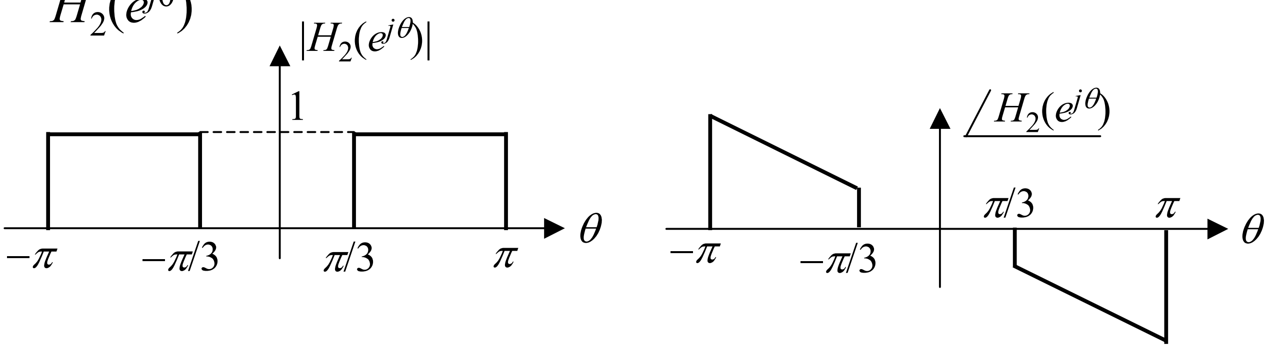
Example:




$H_1(e^{j\theta})$

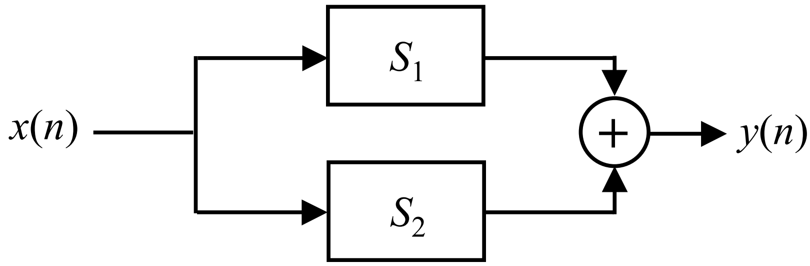



$H_2(e^{j\theta})$

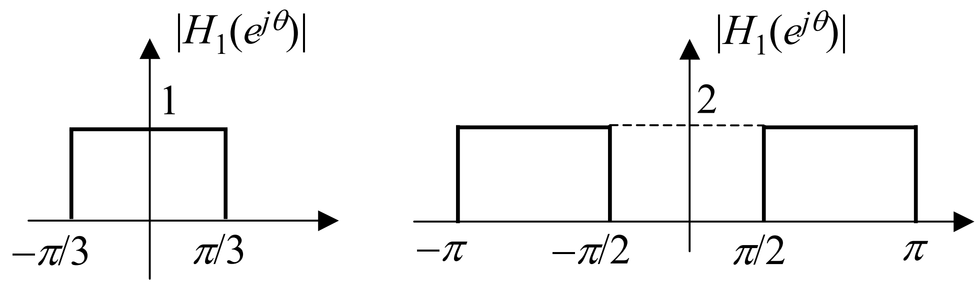



Compute the frequency response to the overall system *S*.

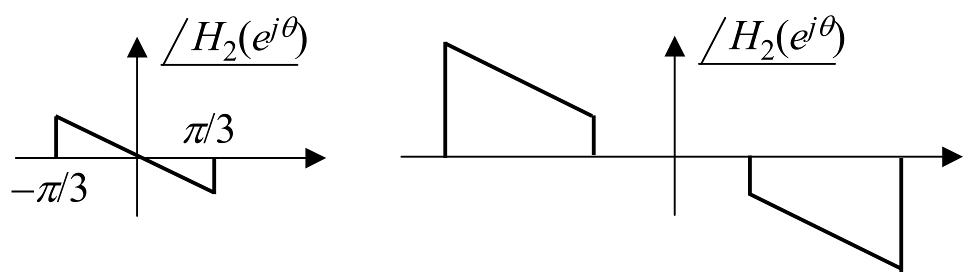
Example:




$H_1(e^{j\theta})$




$H_2(e^{j\theta})$




Compute the frequency response of the overall system.