MA 1043 FINAL Exam Solution

① aug:

$$\begin{array}{c|c} & \left[\begin{array}{cccc|c} 1 & 3 & -2 & 3 & 3 \\ 0 & -1 & 1 & -5 & 4 \\ 1 & 3 & -2 & -1 & 7 \\ -2 & -3 & 1 & 0 & -9 \end{array}\right] \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{array}{l} \times(-1)\,(b) \\ -a\ (1) \\ +2a\ (3) \end{array} \end{array} \sim \begin{array}{c|c} & \left[\begin{array}{ccccc} 1 & 3 & -2 & 3 & 3 \\ 0 & 1 & -1 & 5 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{array}{l} -3c\ (2) \\ -5c\ (1) \end{array} \end{array} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 + x_3 = 3$, $x_1 = 3 - x_3$

$x_2 - x_3 = 1$, $x_2 = 1 + x_3$

$x_4 = -1$

x_3 free

$$\begin{array}{c|c} & \left[\begin{array}{ccccc} c & 0 & 0 & 0 & -4 & 4 \\ c & 0 & 0 & 0 & 1 & -1 \\ d & 0 & 3 & -3 & 6 & -3 \\ d & 0 & 0 & 0 & -9 & 9 \\ d & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ & \begin{array}{l} \times -1/4\ (2) \\ \\ +3b\ (4) \\ +9c\ (5) \\ \\ \end{array} \end{array} \sim \begin{array}{c|c} & \left[\begin{array}{ccccc} b & 0 & 1 & -1 & 0 & 1 \\ a & 1 & 3 & -2 & 0 & 6 \\ a & 1 & 0 & 1 & 0 & 3 \end{array}\right] \\ & \begin{array}{l} \\ -3b\ (3) \\ \\ \end{array} \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - x_3 \\ 1 + x_3 \\ x_3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad (t = x_3)$$

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②

$$A = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} 1 & -3 & -2 \\ -1 & -1 & -3 \\ 3 & -1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 \\ -1 & 7 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \begin{bmatrix} -3 & -2 \end{bmatrix} = \begin{bmatrix} -4 & -5 \\ 8 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 13 \end{bmatrix} - \begin{bmatrix} -2 \end{bmatrix} \begin{bmatrix} -5 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$$

$$- \begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix} \qquad - \begin{bmatrix} 10 \end{bmatrix}$$

$\times(1/1)$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -3 & -2 \\ 0 & & \\ 0 & 0 & \end{bmatrix}$$

$\times(1/-4)$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -3 & -2 \\ 0 & -4 & -5 \\ 0 & 0 & \end{bmatrix}$$

$\times(1/3)$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -3 & -2 \\ 0 & -4 & -5 \\ 0 & 0 & 3 \end{bmatrix}$$

So $A = LU$? $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & -2 \\ 0 & -4 & -5 \\ 0 & 0 & 3 \end{bmatrix}$, then solve $A\vec{x} = \vec{b} = L \underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$

$L\vec{y} = \vec{b} = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$

$U\vec{x} = \vec{y}$

$$L\vec{y} = \vec{b} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ -1 & 1 & 0 & -1 \\ 3 & -2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ +a \\ -3a \end{array} \sim \left[\begin{array}{cccc} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & -2 & 1 & 13 \end{array}\right] \begin{array}{l} \\ \\ +2b \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \vec{y}$$

$$\vec{y} = \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix}$$

$$U\vec{x} = \vec{y} \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -2 & -4 \\ 0 & -4 & -5 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{l} +\frac{2}{3}c \\ +\frac{5}{3}c \\ \times 1/3 \end{array} \sim \left[\begin{array}{cccc} 1 & -3 & 0 & -2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} -\frac{3}{4}b \\ \times -1/4 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \vec{x}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(3) a) $\det(A) = \begin{vmatrix} 6 & 0 & 0 & 2 \\ 8 & 2 & 5 & 2 \\ 9 & 2 & 1 & 3 \\ 3 & 1 & 0 & 1 \end{vmatrix} \overset{-3\times col(4)}{=} \begin{vmatrix} 0 & 0 & 0 & 2 \\ 2 & 2 & 5 & 2 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{vmatrix} = -(2) + \begin{vmatrix} 2 & 2 & 5 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -(2)(+2)\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -(2)(+2)(-1)|1| = (-2)(2)(-1)(1) = 4$

b) since $\det(A) \neq 0$ then A is invertible so $\lambda = 0$ is NOT an eigenvalue

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(4) $S = \{ f(x) \in C^3(-\infty,\infty) \mid f''' = e^x f(x) \ \& \ f'(3) = 0 \}$, is S a subspace?

check closure of addition: suppose $f \in S$ & $g \in S$, what about $(f+g)$?

$(f+g)''' = f''' + g''' = e^x f(x) + e^x g(x) = e^x (f+g)(x)$ so $(f+g)$ satisfies 1st equation

$(f+g)'(3) = f'(3) + g'(3) = 0 + 0 = 0$ so $(f+g)$ also satisfies 2nd equation, so $(f+g) \in S$ ✓

check closure of scaling: suppose $f \in S$ & c is any scalar (real number), what about (cf)?

$(cf)''' = cf''' = c(e^x f(x)) = e^x c f(x) = e^x (cf)(x)$ so (cf) satisfies 1st eq.

$(cf)'(3) = c f'(3) = c \cdot 0 = 0$, so (cf) also satisfies 2nd eq, so $(cf) \in S$ ✓

This shows that S is closed with respect to vector addition & scaling, so S IS a subspace

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(5) $A = \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1 & -1 & -2 \\ -2 & 2 & 4 \\ 0 & 0 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{array}{l} +2c\ (4) \\ +2a\ (1) \\ \times 1/3\ (3) \\ +a\ (2) \end{array} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$

$\begin{array}{c} b \\ d \\ c \\ a \end{array}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ SWAP (5)

b) basis for row space of A is pivot rows of B: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) basis for column space of A is pivot columns of A: $\rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \\ 2 \end{bmatrix} \right\}$

d) solve $A\vec{x} = \vec{0}$: $\left.\begin{array}{l} x_1 - x_2 = 0 \rightarrow x_1 = x_2 \\ x_3 = 0 \\ x_2 \text{ free} \end{array}\right\}$ $\vec{x} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, basis for null space of A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

e) suppose $\vec{x} \in$ null space & $\vec{y} \in$ row space, so $\vec{x} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{y} = r\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

then $\langle \vec{x}, \vec{y} \rangle = \left\langle t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, r\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle = tr\left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle + ts\left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle = tr(1\cdot 1 + 1(-1) + 0\cdot 0) + ts(1\cdot 0 + 1\cdot 0 + 0\cdot 1)$

$= 0$ so orthogonal

f) $\text{rank}(A) = \#\text{pivots} = 2$, $\text{nullity}(A) = \#\text{free vars} = 1 = \#\text{cols} - \text{rank}$

g) $\text{rank}(A^T) = \text{rank}(A) = 2$, $\text{nullity}(A^T) = (\#\text{rows of } A) - \text{rank} = 4 - 2 = 2$

(6) P = space of polynomials [a wholly owned subsidiary of $C^\infty(-\infty,\infty)$]

define inner product as $\langle p(x), q(x)\rangle = \int_0^1 p(x)\,q(x)\,dx$

given $p(x)=1$, $q(x)=x$, then $\langle p,q\rangle = \int_0^1 1\cdot x\,dx = \left[\frac{x^2}{2}\right]_0^1 = 1/2$

and $\langle p,p\rangle = \int_0^1 1\cdot 1\,dx = [x]_0^1 = 1$ and $\langle q,q\rangle = \int_0^1 x\cdot x\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3$

a) so if θ is the angle betw p & q, $\cos\theta = \frac{\langle p,q\rangle}{\|p\|\,\|q\|} = \frac{(1/2)}{\sqrt{1}\sqrt{1/3}} = \frac{\sqrt{3}}{2}$, so $\theta = \frac{\pi}{6} = 30°$

b) $\text{proj}_q p = \frac{\langle p,q\rangle}{\langle q,q\rangle} q = \frac{(1/2)}{(1/3)} q = \frac{3}{2}x = w(x)$

c) $r(x) = p(x) - w(x) = 1 - \frac{3}{2}x$, then $\langle r(x), q(x)\rangle = \int_0^1 \left(1-\frac{3}{2}x\right)(x)\,dx = \int_0^1 x - \frac{3}{2}x^2\,dx = \left[\frac{x^2}{2} - \frac{x^3}{2}\right]_0^1 = \frac{1}{2}-\frac{1}{2} = 0$ ✓ orthogonal

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(7) $A = \begin{bmatrix} -4 & 6 & 0 \\ 0 & -1 & -3 \\ -6 & 9 & 5 \end{bmatrix}$. find e-vals: $\det(A-\lambda I) = \begin{vmatrix} -4-\lambda & 6 & 0 \\ 0 & -1-\lambda & -3 \\ -6 & 9 & 5-\lambda \end{vmatrix} = (-4-\lambda)\left[(-1-\lambda)(5-\lambda)-(-3)9\right] - 6\left[0-(-3)(-6)\right] + 0$

$= (-4-\lambda)\left[\lambda^2-4\lambda+22\right] - 6[-18]$

$\det(A-\lambda I) = -\left[\lambda^3+6\lambda+88-108\right] = -\left[\lambda^3+6\lambda-20\right] = 0$ so $\lambda^3+6\lambda-20=0$

try integer factors of 20: $+1$: $1^3+6\cdot 1-20 = -13 \neq 0$, -1: $(-1)^3+6(-1)-20 = -27 \neq 0$, $+2$: $2^3+6\cdot 2-20 = 8+12-20 = 0$ ✓

so $\lambda_1 = 2$, factor out $(\lambda-2)$

$$\begin{array}{r} \lambda^2+2\lambda+10 \\ \lambda-2\,\overline{)\,\lambda^3 + 6\lambda - 20} \\ \underline{\lambda^3-2\lambda^2} \\ 0+2\lambda^2+6\lambda-20 \\ \underline{2\lambda^2-4\lambda} \\ 0+10\lambda-20 \end{array}$$

, so $\lambda^2+2\lambda+10=0$ for other roots: $\lambda = \frac{-2\pm\sqrt{2^2-4\cdot 1\cdot 10}}{2\cdot 1} = \frac{-2\pm\sqrt{-36}}{2}$

or $\lambda = -1\pm 3i$

eigenvalues: $\boxed{2,\ -1+3i,\ -1-3i}$

$\lambda_1 = 2$ solve $(A-2I)\vec{x}=0$: $\begin{bmatrix} -6 & 6 & 0 \\ 0 & -3 & -3 \\ -6 & 9 & 3 \end{bmatrix} \begin{matrix} \times -1/6 \\ \times -1/3 \\ -a+b \end{matrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + b \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\left.\begin{matrix} x_1+x_3=0 \\ x_2+x_3=0 \\ x_3 \text{ free} \end{matrix}\right\}$ $\vec{x}_1 = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ eigenspace of $\lambda_1 = 2$

$\lambda_2 = -1+3i$ solve $\underbrace{(A-(-1+3i)I)}_{A+(1-3i)I}\vec{x}=\vec{0}$

$$\begin{array}{c|ccc|l} a & -3-3i & 6 & 0 & +(3+3i)\times c\ (4) \\ b & 0 & -3i & -3 & \times i/3\ (1) \\ c & -6 & 9 & 6-3i & -9b\ (2) \end{array} \sim \begin{bmatrix} 1 & 0 & -1-i \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - (1+i)x_3 = 0$ $\quad x_1 = (1+i)x_3$

$x_2 - i x_3 = 0$ $\quad x_2 = i x_3$

x_3 free

$$\begin{array}{c|ccc|l} b & 0 & 1 & -i & \\ e & -6 & 0 & 6+6i & \times -1/6\ (3) \\ c & 1 & 0 & -1-i & \\ \text{for step 4: } (3+3i)c & 3+3i & 0 & -3+3-3i-3i = -6i & \\ a & 0 & 6 & -6i & -6b\ (5) \\ a & 0 & 0 & 0 & \end{array}$$

(6) swap a & c

$\vec{x}_2 = \begin{bmatrix} (1+i)x_3 \\ i\,x_3 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1+i \\ i \\ 1 \end{bmatrix}$ eigenspace of $\lambda_2 = -1+3i$

so $\vec{x}_3 = r\begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix}$ take conjugate

eigenspace for $\lambda_3 = -1-3i$