5 LYAPUNOV STABILITY

The concept of stability according to Lyapunov has found many applications in control systems; in fact the whole theory of dynamical systems is based, to a great extent, on Lyapunov's methods.

5.1 Lyapunov Functions

Consider the nonlinear system

$$\dot{x} = f(x) \ .$$

Let an equilibrium point of the system be $\overline{x}$,

$$f(\overline{x}) = 0 \ .$$

We say that $\overline{x}$ is stable in the sense of Lyapunov if there exists a positive quantity ϵ such that for every $\delta = \delta(\epsilon)$ we have

$$|x(t_0) - \overline{x}| < \delta \Longrightarrow |x(t) - \overline{x}| < \epsilon \ ,$$

for all $t > t_0$. We say that $\overline{x}$ is asymptotically stable if it is stable and,

$$|x(t) - \overline{x}| \to 0 \quad \text{as} \quad t \to \infty \ .$$

We cal $\overline{x}$ unstable if it is not stable.

The question, of course, is: How do we determine stability or instability of $\overline{x}$? Lyapunov introduced two main methods:

The first is called Lyapunov's *first* or *indirect* method: we have already seen it as the linearization technique. Start with a nonlinear system

$$\dot{x} = f(x) \ .$$

Expand in Taylor series around $\overline{x}$ (we also redefine $x \to x - \overline{x}$),

$$\dot{x} = Ax + g(x) \ ,$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} \ ,$$

is the Jacobian matrix of $f(x)$ evaluated at $\overline{x}$, and $g(x)$ contains the higher order terms; i.e.,

$$\lim_{|x| \to 0} \frac{|g(x)|}{|x|} = 0 \ .$$

Then, the nonlinear system $\dot{x} = f(x)$ is asymptotically stable if and only if the linear system $\dot{x} = Ax$ is; i.e., if all eigenvalues of A have negative real parts. This method is very popular because it is easy to apply and it works well for most systems, all we need to do is to be able to evaluate partial derivatives. One disadvantage of the method is that if some eigenvalues of A are zero and the rest have negative real parts, then we cannot draw any conclusions on the nonlinear system, the equilibrium $\overline{x}$ can be either stable or unstable. The major drawback of the method, however, is that since it involves linearization it is applied for situations when the initial conditions are "close" to the equilibrium $\overline{x}$. The method provides no indication as to how close is "close", this is something which may be extremely important in practical applications.

The second method is Lyapunov's *second* or *direct* method: this is a generalization of Lagrange's concept of stability of minimum potential energy. Consider the nonlinear system $\dot{x} = f(x)$. Suppose that there exists a function, called Lyapunov function, $V(x)$ with the following properties:

1. $V(\overline{x}) = 0$.
2. $V(x) > 0$, for $x \neq \overline{x}$.
3. $\dot{V}(x) < 0$ along trajectories of $\dot{x} = f(x)$.

Then, $\overline{x}$ is asymptotically stable. We can see that the method hinges on the existence of a Lyapunov function, which is an energy–like function, zero at equilibrium, positive definite everywhere else, and continuously decreasing as we approach the equilibrium. It should be noted that the derivative $\dot{V}(x)$ is understood as the total differential along solution curves of $\dot{x} = f(x)$; i.e.,

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} \\
&= \frac{\partial V}{\partial x} f(x) \\
&= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} x_2 + \cdots + \frac{\partial V}{\partial x_n} f_n \ .
\end{aligned}$$

The method is very powerful and it has several advantages:

- answers questions of stability of nonlinear systems,
- can easily handle time varying systems $\dot{x} = f(x, t)$,
- can determine asymptotic stability as well as plain stability,
- can determine the region of asymptotic stability or the domain of attraction of an equilibrium.

As an example, consider an oscillator with a nonlinear spring:

$$\ddot{y} + 3\dot{y} + y^3 = 0 \ .$$

If we were to linearize this system we would get $\ddot{y}+3\dot{y}=0$, which has characteristic equation $s(s+3)=0$. The -3 characteristic root corresponds to the damping term but notice the existence of a 0 root from the lack of a linear term in the spring restoring force. The linearized version of the system cannot recognize the existence of a nonlinear spring term and it fails to produce a non–zero characteristic root related to the restoring force. To see if this nonlinear spring produces a stable or unstable system we have to resort to Lyapunov functions. The state space form of the system is

$$\begin{aligned} \dot{x}_1 &= x_2 \,, \\ \dot{x}_2 &= -3x_2 - x_1^3 \,, \end{aligned}$$

with equilibrium $\overline{x}_1 = \overline{x}_2 = 0$. Let's try for a Lyapunov function

$$V(x) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4 \,.$$

We can see that $V(x) > 0$ for all x_1, x_2. The time derivative of V is

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= x_1^3 x_2 + x_2(-3x_2 - x_1^3) \\ &= -3x_2^2 \\ &< 0 \,. \end{aligned}$$

It follows then that $\overline{x}$ is asymptotically stable.

The main drawback of the method is that there is no systematic way of obtaining Lyapunov functions, this is more of an art than science. For simple second order systems (like the one above) a good selection for a Lyapunov function is the total energy of the system (kinetic plus potential energy). Also, it is always possible to find a Lyapunov function for a linear system in the form

$$\dot{x} = Ax \,.$$

Choose as Lyapunov function the quadratic form

$$V(x) = x^T P x \,,$$

where P is a symmetric positive definite matrix. Then we have

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x \\ &= -x^T Q x \,, \end{aligned}$$

where we have denoted

$$A^T P + PA = -Q \,.$$

If the matrix Q is positive definite, then the system is asymptotically stable. Therefore, we could pick $Q = I$, the identity matrix, and solve

$$A^T P + PA = -I \ ,$$

for P and see if P is positive definite (we can do this by looking at the n principal minors of P — Sylvester's criterion). The equation

$$A^T P + PA = -Q \ ,$$

is called Lyapunov's matrix equation and its solution is easy through MATLAB by using the command `lyap`. Of course one could argue that having an equation to determine a Lyapunov function for linear systems is useless; after all for a linear system we can always look at the eigenvalues of A to determine stability or instability. This is true, the usefulness of Lyapunov's matrix equation for linear systems is that it can provide an initial estimate for a Lyapunov function for a nonlinear system in cases where this is done computationally. Furthermore, it can be used to show stability, as we will see in the next section, of the linear quadratic regulator design.

5.2 Examples

We present three examples here that demonstrate three important applications of Lyapunov's method, namely

1. How to assess the importance of nonlinear terms in stability or instability.
2. How to estimate the domain of attraction of an equilibrium point.
3. How to design a control law that guarantees global asymptotic stability; i.e., with infinitely large domain of attraction, for a nonlinear system.

All of the above problems are very difficult, in general, and we shouldn't think that we can easily generalize the relatively simple examples we present here.

As our **first example**, suppose we have the system

$$\begin{aligned} \dot{x}_1 &= -x_2 + a x_1 x_2^2 \ , \\ \dot{x}_2 &= +x_1 - b x_1^2 x_2 \ , \end{aligned}$$

with $a \neq b$. To find the equilibrium of the system we have to solve

$$\begin{aligned} -\overline{x}_2 + a\overline{x}_1\overline{x}_2^2 &= 0 \ , \\ +\overline{x}_1 - b\overline{x}_1^2\overline{x}_2 &= 0 \ . \end{aligned}$$

Multiplying the first equation by $\overline{x}_1$, the second by $\overline{x}_2$ and adding we get

$$\overline{x}_1^2\overline{x}_2^2(a-b) = 0 \ ,$$

from which $\overline{x}_1 = 0$ or $\overline{x}_2 = 0$. If $\overline{x}_1 = 0$ then we see from the first equation that $\overline{x}_2 = 0$ as well, and similarly if we assume that $\overline{x}_2 = 0$. Therefore, the unique equilibrium of the system is $\overline{x}_1 = \overline{x}_2 = 0$. The linearized system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

The characteristic equation is

$$\det \begin{vmatrix} -s & -1 \\ 1 & -s \end{vmatrix} = 0 \implies s^2 + 1 = 0 \implies s = \pm \omega i .$$

Since the characteristic roots are purely imaginary, we cannot draw any conclusion on the stability of the nonlinear system. We have to resort to Lyapunov functions. Let's try for $V(x)$ the sum of the "kinetic" and "potential" energy of the linear system (this doesn't always work of course), we get

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 .$$

We see that $V(x) > 0$ for all x_1, x_2. Then

$$\begin{aligned} \dot{V}(x) &= x_1(-x_2 + a x_1 x_2^2) + x_2(x_1 - b x_1^2 x_2) \\ &= -x_1 x_2 + a x_1^2 x_2^2 + x_1 x_2 - b x_1^2 x_2^2 \\ &= (a-b) x_1^2 x_2^2 . \end{aligned}$$

Therefore, we see that

$$\begin{aligned} &\text{if} \quad a < b \implies \text{the system is asymptotically stable ,} \\ &\text{if} \quad a > b \implies \text{the system is unstable ,} \end{aligned}$$

a result which could not have been obtained by linearization.

As our **second example**, suppose we want to determine the stability of the origin $(0,0)$ of the nonlinear system (show that this is the equilibrium of the system),

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + x_1(x_1^2 + x_2^2) , \\ \dot{x}_2 &= -x_1 - x_2 + x_2(x_1^2 + x_2^2) . \end{aligned}$$

The easiest way to show stability is by linearization. The linearized form of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

The characteristic equation of the system is

$$s^2 + 2s + 2 = 0 ,$$

and we can see that the system is stable, the roots of the characteristic equation have negative real parts. Now since this result is based on linearization, it says that if the initial condition is “close” to the equilibrium point $(0,0)$ then the solution will tend to the equilibrium as $t \to \infty$. To find how close is “close” we need to get an estimate of the *domain of attraction*. We can do this by using Lyapunov theory. Let’s try a Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \; .$$

Form

$$\begin{aligned}
\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1(-x_1 + x_2 + x_1^3 + x_1x_2^2) + x_2(-x_1 - x_2 + x_2x_1^2 + x_2^3) \\
&= -x_1^2 + x_1x_2 + x_1^4 + x_1^2x_2^2 - x_1x_2 - x_2^2 + x_2^2x_1^2 + x_2^4 \\
&= x_1^4 + x_2^4 + 2x_1^2x_2^2 - x_1^2 - x_2^2 \\
&= (x_1^2 + x_2^2)^2 - (x_1^2 + x_2^2) \\
&= (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \; .
\end{aligned}$$

We can see, therefore, that stability is guaranteed if

$$\dot{V}(x) < 0 \quad \text{or} \quad x_1^2 + x_2^2 < 1 \; ,$$

which means that the domain of attraction of the equilibrium is a circular disk of radius 1. As long as the initial conditions are inside this disk, it is guaranteed that the solution will end up at the stable equilibrium. In case where the initial conditions lie outside the disk then convergence is not guaranteed. It should be mentioned that the above disk is an estimate of the domain of attraction based on the particular Lyapunov function we selected. A different Lyapunov function could heve produced a different estimate of the domain of attraction.

As our **third example**, consider the motion of a space vehicle about the principal axes of inertia. The Euler equations are

$$\begin{aligned}
A\dot{\omega}_x - (B - C)\omega_y\omega_z &= T_x \; , \\
B\dot{\omega}_y - (C - A)\omega_z\omega_x &= T_y \; , \\
C\dot{\omega}_z - (A - B)\omega_x\omega_y &= T_z \; ,
\end{aligned}$$

where A, B, and C denote the moments of inertia about the principal axes, ω_x, ω_y, and ω_z denote the angular velocities about the principal axes; and T_x, T_y, T_z are the control torques. Assume that the space vehicle is tumbling in orbit. It is desired to stop the tumbling by applying control torques which are assumed to be

$$\begin{aligned}
T_x &= k_1 A\omega_x \; , \\
T_y &= k_2 B\omega_y \; , \\
T_z &= k_3 C\omega_z \; ,
\end{aligned}$$

where k_1, k_2, k_3 are the feedback gains. The unique equilibrium of the system is $\omega_x = \omega_y = \omega_z = 0$. If we substitute the equations for the control torques we get the closed loop system

$$\dot{\omega}_x = \frac{B - C}{A}\omega_y\omega_z + k_1\omega_x \; ,$$

$$\dot{\omega}_y = \frac{C-A}{B}\omega_z\omega_x + k_2\omega_y \ ,$$
$$\dot{\omega}_z = \frac{A-B}{C}\omega_x\omega_y + k_3\omega_z \ .$$

If we linearize the system around its equilibrium we have

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \ .$$

We can see that the eigenvalues of the closed loop matrix are the same as the feedback gains k_1, k_2, k_3. Therefore, for stability we want negative poles and, as a result, we select negative gains k_1, k_2, k_3 for the three control torques. So far we have used linear methods. What we are really interested though is the following: will the above gain selection guarantee globally stable operation of the system? In other words, will our control law be able to stop the vehicle tumbling for *any* set of initial conditions? To see this we have to resort to Lyapunov methods. Choose as our Lyapunov function

$$V(\omega) = \frac{1}{2}A\omega_x^2 + \frac{1}{2}B\omega_y^2 + \frac{1}{2}C\omega_z^2 \ ,$$

which is the total kinetic energy of the vehicle. We see that V is positive definite, and its time derivative is

$$\dot{V}(\omega) = k_1 A\omega_x^2 + k_2 B\omega_y^2 + k_3 C\omega_z^2 \ ,$$

which is always negative if the gains are selected negative. Therefore, the above gain selection guarantees stability of the nonlinear system regardless of the initial conditions.

5.3 Sliding Mode Control

As an application of Lyapunov method, consider a single input system linear in the control effort

$$\dot{x} = f(x) + g(x)u \ ,$$

where $f(x)$, $g(x)$ are, in general, nonlinear functions in x. We want to design u such that we guarantee stability of $x = 0$.

Choose the Lyapunov function

$$V(x) = \frac{1}{2}\left[\sigma(x)\right]^2 \ ,$$

where

$$\sigma(x) = s^T x \ .$$

The scalar function $\sigma(x)$ can be viewed as a weighted sum of the errors in the states x. For stability, we want the time derivative of $V(x)$ to be negative,

$$\dot{V}(x) = \sigma\dot{\sigma} < 0 \ ,$$

which can be achieved if

$$\sigma\dot{\sigma} = -\eta^2|\sigma| \ ,$$

which means that

$$\dot{\sigma} = -\eta^2 \mathrm{sign}(\sigma) \ ,$$

where

$$\mathrm{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0 \ , \\ -1 & \text{if } \sigma < 0 \ . \end{cases}$$

Using $\sigma(x) = s^T x$, we get

$$\dot{\sigma} = s^T \dot{x} = s^T f(x) + s^T g(x) u = -\eta^2 \mathrm{sign}(\sigma) \ ,$$

and solving for u we get the control law

$$u = -\left[s^T g(x)\right]^{-1} s^T f(x) - \left[s^T g(x)\right]^{-1} \eta^2 \mathrm{sign}(\sigma) \ .$$

We can see that this control law is the sum of two terms. The first term is a nonlinear state feedback, and the second term is a switching control law. The term η^2 is an arbitrary positive quantity, we usually select it such that $\dot{V}$ is negative even in the presence of modeling errors and disturbances.

The above control law guarantees stability of $\sigma(x) = 0$, or $s^T x = 0$. We need to find s such that stability of $x = 0$ is guaranteed. If $\sigma(x) = 0$, the system becomes

$$u = -\left[s^T g(x)\right]^{-1} s^T f(x) \ ,$$

and

$$\dot{x} = f(x) - g(x) \left[s^T g(x)\right]^{-1} s^T f(x) \ .$$

If we linearize this system,

$$A = \left.\frac{\partial f}{\partial x}\right|_0 \ , \quad b = g(0) \ ,$$

we get a linear system

$$\dot{x} = Ax + bu \ .$$

Then, on $\sigma(x) = 0$ we have

$$\begin{aligned} \dot{x} &= Ax - b(s^T b)^{-1} s^T A x \\ &= \left[A - b(s^T b)^{-1} s^T A\right] x \ . \end{aligned}$$

The closed loop dynamics matrix is

$$A_C = A - b \underbrace{(s^T b)^{-1} s^T A}_{k} = A - bk \ .$$

Then

$$k = (s^T b)^{-1} s^T A \Longrightarrow s^T b k = s^T A \Longrightarrow s^T A - s^T b k = 0 \ ,$$

or

$$s^T (A - bk) = 0 \Longrightarrow (A - bk)^T s = 0 \Longrightarrow A_C^T s = 0 \Longrightarrow (A_C^T - 0 \cdot I) = 0 \ .$$

We see then that s is the eigenvector of A_C^T that corresponds to the zero eigenvalue. The design procedure, therefore, can be summarized as follows:

- Pole placement of $A - bk$, specify one eigenvalue to be zero and the rest negative. Find k and therefore, find $A_C = A - bk$.
- Find s from $A_C^T s = 0$. Set $\sigma = s^T x$.
- Implement the control law

  $$u = -\left[s^T g(x)\right]^{-1} s^T f(x) - \left[s^T g(x)\right]^{-1} \eta^2 \text{sign}(\sigma) \ ,$$

  if we have a nonlinear system, or

  $$u = -(s^T b)^{-1} s^T A x - (s^T b)^{-1} \eta^2 \text{sign}(\sigma) \ ,$$

  if we have a linear system.