3 OBSERVER DESIGN

So far we have developed the means to establish a control law; i.e, software which commands a certain action from the system actuators. What is needed is the state x. In reality, however, what is available to us from hardware is the output y through a set of sensors. In order to complete the picture, therefore, we need to estimate x given y.

3.1 State Estimators

Say we have the system

$$\dot{x} = Ax + Bu \ ,$$

and we want to use a control

$$u = -Kx \ .$$

Suppose, however, that we only have the measurements (output)

$$\underbrace{y}_{p\times 1} = \underbrace{C}_{p\times n}\underbrace{x}_{n\times 1} \ , \quad p < n \ ,$$

instead of x. Note that if p were equal to n then we could use $x = C^{-1}y$ and our troubles would be over; the interesting case is when we have less sensors available than the number of states, $p < n$. It may be undesirable, expensive, or impossible to directly measure all of the states. What we can do is to dynamically use the p measurements to estimate all the states in x. If we denote the estimate of the state x as $\widehat{x}$, the error in that estimate will be

$$\underbrace{\widetilde{x}}_{\text{error}} = \underbrace{x}_{\text{actual}} - \underbrace{\widehat{x}}_{\text{estimate}} \ .$$

Then we could feed back this estimate $\widehat{x}$ in place of the actual state; i.e.,

$$u = -K\widehat{x} \ .$$

What we need now is to construct a state estimator or observer. Consider feeding back the difference between the estimated and measured outputs and correcting the model continuously with this error signal

$$\dot{\widehat{x}} = A\widehat{x} + Bu + L(y - C\widehat{x}) \ ,$$

where

$$\begin{aligned}
A\widehat{x} + Bu \ &: \ \text{system model, } \widehat{x} \text{ should behave like } x \ , \\
L \ &: \ \text{observer gain matrix, to be determined} \ , \\
y \ &: \ \text{actual measurement} \ , \\
C\widehat{x} \ &: \ \text{measurement if } x \text{ were } \widehat{x} \ .
\end{aligned}$$

In order to establish L we can consider the dynamics of the error in the estimate,

$$\begin{aligned}
\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} \Longrightarrow \\
\dot{\tilde{x}} &= Ax + Bu - A\hat{x} - Bu - L(y - C\hat{x}) \Longrightarrow \\
\dot{\tilde{x}} &= A(x - \hat{x}) - L(Cx - C\hat{x}) \Longrightarrow \\
\dot{\tilde{x}} &= (A - LC)\tilde{x} \; .
\end{aligned}$$

The error in the estimate will be determined by the eigenvalues of $[A - LC]$ which we can obtain from $\det[A - LC - sI] = 0$. If (A, C) is observable, we can pick the elements of L to give the error arbitrary dynamics, similarly to the control design. We should choose the eigenvalues of $[A - LC]$ to be further to the left in the s–plane than the eigenvalues of $[A - BK]$. Then the error in the estimate will die quickly compared to the dynamics of the system.

The combined controller and observer equations are

$$\begin{aligned}
\dot{x} &= Ax - BK\hat{x} \; , \\
\dot{\hat{x}} &= LCx + (A - LC - BK)\hat{x} \; , \\
y &= Cx \; ,
\end{aligned}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \; ,$$

and

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \; .$$

In block diagram form this appears as shown in Figure 17.

If we use

$$u = -K\hat{x} = -K(x - \tilde{x}) \; ,$$

we get

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \; ,$$

which has the following characteristic equation

$$\det[A - BK - sI] \; \cdot \det[A - LC - sI] = 0 \; .$$

This indicates that the dynamics of the observer are completely independent of the dynamics (eigenvalues) of the controller. Thus, K and L can be designed separately.

3.2 Duality

Remember the controller design for $\dot{x} = Ax + Bu$, $y = Cx$ by placing the eigenvalues of $[A - BK]$. For the observer design we want to place the eigenvalues of $[A - LC]$. But the

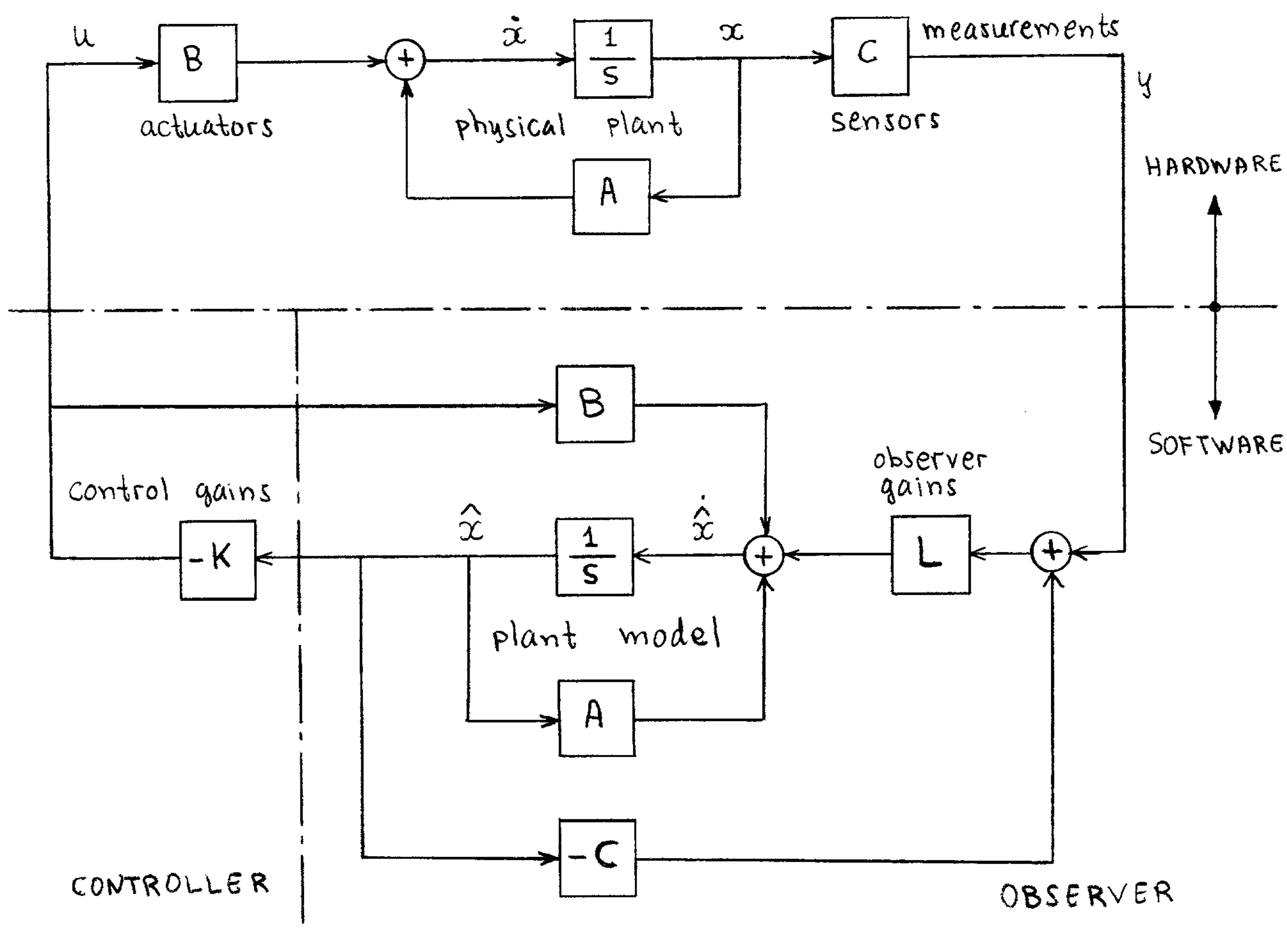

Figure 1: Compensator block diagram

eigenvalues of $[A - LC]$ are the same as the eigenvalues of $[A - LC]^T$ and these are the same as the eigenvalues of $[A^T - C^T L^T]$. Therefore, instead of designing an observer for the system $\dot{x} = Ax + Bu$, $y = Cx$ we can design a controller for $\dot{x} = A^T + C^T u$. This is the duality principle between controller and observer,

$$\begin{array}{ccc} \text{controller} & \longleftrightarrow & \text{observer} \\ A & \longleftrightarrow & A^T \\ B & \longleftrightarrow & C^T \\ C & \longleftrightarrow & B^T \end{array}$$

For any system

$$\begin{aligned} \dot{x} &= Ax + Bu \, , \\ y &= Cx \, , \end{aligned}$$

its dual system is

$$\begin{aligned} \dot{x} &= A^T x + C^T u \, , \\ y &= B^T x \, . \end{aligned}$$

The controllability matrix of a system is the observability matrix of its dual and vice versa. If in the observer canonical form, starting from the output, all signal flows are reversed — summers are changed to nodes and nodes are changed to summers — we obtain the control canonical form.

3.3 Pole Placement for Single Output Systems

When there is only one output variable, the output equation is

$$y = \left[\begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] .$$

Thus, C is a row vector

$$C = \left[\begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \right] ,$$

and the observer gain matrix L is a column vector

$$L = \left[\begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} \right] .$$

Now recall the expression we had for the controller gain matrix

$$K^T = \left[(\mathcal{C}W)^T \right]^{-1} (-a + \alpha) \, .$$

By duality, the observer gain matrix must be

$$L = \left[(\mathcal{O}W)^T \right]^{-1} (-a + \alpha) \ ,$$

where

$$\begin{aligned} \mathcal{O} &= \text{observability matrix} \ , \\ a &= \text{coefficients of original characteristic equation} \ , \\ \alpha &= \text{coefficients of desired characteristic equation} \ . \end{aligned}$$

The presence of more than one outputs provides more flexibility; it is possible to place all the eigenvalues and do other things too. Or, alternatively, some of the observer gains can be set to zero to simplify the resulting observer structure.

3.4 Compensator Design

Recall that the eigenvalues of the controller were not affected by the eigenvalues of the observer, this allows us to design the controller and observer separately which is known as the *separation principle*. The combination is called a *compensator*,

$$(\text{controller}) + (\text{estimator}) = (\text{compensator}) \ .$$

For the system

$$\begin{aligned} \dot{x} &= Ax + Bu \ , \\ y &= Cx \ , \end{aligned}$$

we have the controller

$$u = -Kx \ ,$$

the observer

$$\dot{\widehat{x}} = A\widehat{x} + Bu + L(y - C\widehat{x}) \ ,$$

and, using the separation principle, we can write

$$u = -K\widehat{x} \ .$$

The block diagram of the compensator is shown in Figure 17.

Using the above equations we get

$$\begin{aligned} \dot{x} &= Ax - BK\widehat{x} \\ &= Ax - BK(x - \widetilde{x}) \\ &= (A - BK)x + BK\widetilde{x} \\ &= A_C x + BK\widetilde{x} \ , \end{aligned}$$

and

$$\dot{\widehat{x}} = A\widehat{x} - BK\widehat{x} + L(Cx - C\widehat{x}) \ .$$

Therefore,

$$\dot{\widetilde{x}} = \dot{x} - \dot{\widehat{x}} = (A - LC)\widetilde{x} = \widehat{A}\widetilde{x} \ .$$

Taking Laplace transforms,

$$\begin{aligned} (sI - A_C)x(s) &= BK\widetilde{x}(s) + x(t_0) \ , \\ (sI - \widehat{A})\widetilde{x}(s) &= \widetilde{x}(t_0) \Longrightarrow \widetilde{x}(s) = (sI - \widehat{A})^{-1}\widetilde{x}(t_0) \ . \end{aligned}$$

Therefore,

$$\begin{aligned} x(s) &= (sI - A_C)^{-1}BK\widetilde{x}(s) + (sI - A_C)^{-1}x(t_0) \ , \\ &= (sI - A_C)^{-1}BK(sI - \widehat{A})^{-1}\widetilde{x}(t_0) + (sI - A_C)^{-1}x(t_0) \ , \end{aligned}$$

and we can see that the transient response of the state is the sum of two part: one part due to the initial estimation error $\widetilde{x}(t_0)$, and one part due to the initial state $x(t_0)$.

In order to obtain the transfer function of the compensator, we have

$$\dot{\widehat{x}} = (A - BK - LC)\widehat{x} + Ly \ ,$$

or

$$\widehat{x}(s) = (sI - A + BK + LC)^{-1}Ly(s) \ .$$

Then

$$u(s) = -K\widehat{x}(s) = -K(sI - A + BK + LC)^{-1}Ly(s) \ .$$

The transfer function of the compensator, $D(s)$, is defined between plant output and plant input by

$$u(s) = -D(s)y(s) \ ,$$

so

$$\begin{aligned} D(s) &= K(sI - A + BK + LC)^{-1}L \\ &= K(sI - \widehat{A}_C)^{-1}L \ , \end{aligned}$$

where

$$\widehat{A}_C = A - BK - LC = A_C - LC = \widehat{A} - BK \ .$$

We can define the following:

- compensator poles = zeros of $|sI - \widehat{A}_C|$,
- open loop plant poles = zeros of $|sI - A|$,
- controller poles = zeros of $|sI - A_C|$,
- observer poles = zeros of $|sI - \widehat{A}|$.

All of the above are, in general, different. If $\widehat{A}$ and A_C are chosen independently, it may even happen that $\widehat{A}_C$ has roots in the right half s–plane, which means that even though the complete system is still stable, we can get an "unstable" compensator. This is not catastrophic, the main serious consequence of an unstable compensator is that the closed loop system will only be conditionally stable and, therefore, may not be very robust with respect to unmodeled dynamics and parameter variations.

In summary, the compensator design proceeds as follows:

1. Design a control law assuming that all states are available.
2. Design an observer to estimate the (missing) states.
3. Combine the full state control law with the observer to obtain the compensator design.

Example: Consider the submarine pitch angle control developed in the previous section. With poles at -0.3, and if not all states θ, w, q are directly measurable, we have to use

$$\delta = -(-0.8451\widehat{\theta} - 1.4733\widehat{w} + 0.9807\widehat{q}) \ .$$

Assume, however, that the only sensor we have is a rate gyro that measures the pitch rate q. We have to design an observer to estimate θ, w, q, using the q measurements. First, is this possible? To do this we have to check the observability of the system. The output equation is

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ w \\ q \end{bmatrix} ,$$

and the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1.0000 \\ -0.0360 & 0.1260 & -0.7395 \\ 0.0283 & -0.1338 & 0.4214 \end{bmatrix} ,$$

which has rank 3; i.e., the system *is* observable. In order to design the observer gains we use the duality principle and we issue the MATLAB command `place` which we already used for the controller: here we use A' instead of A and C' instead of B (the prime in MATLAB signifies a transpose). The observer poles are selected, say at -0.6, -0.61, -0.62; these are twice as negative as the controller poles so the error in the estimate should die out faster than the system dynamics. The observer gains are

$$L = \begin{bmatrix} -21.9614 \\ -2.2636 \\ 0.7685 \end{bmatrix} ,$$

and the observer equations

$$\dot{\widehat{x}} = A\widehat{x} + Bu + L(y - C\widehat{x}) \ ,$$

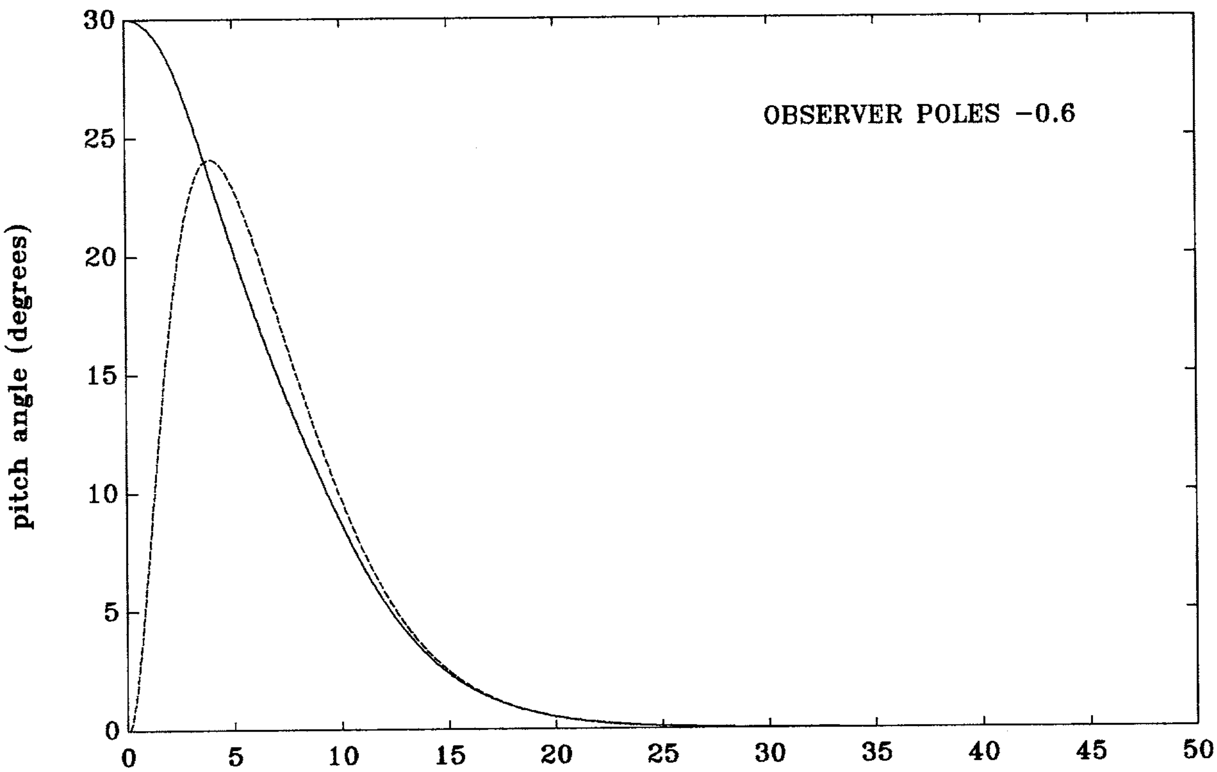

Figure 2: Compensator response for fast observer poles

or, if we substitute the values for A, B, C, and L,

$$\begin{aligned}
\dot{\widehat{\theta}} &= \widehat{q} - 21.9614(q - \widehat{q}) \;, \\
\dot{\widehat{w}} &= 0.0135\widehat{\theta} - 0.322\widehat{w} - 0.7102\widehat{q} + 0.0322\delta - 2.2636(q - \widehat{q}) \;, \\
\dot{\widehat{q}} &= -0.036\widehat{\theta} + 0.126\widehat{w} - 0.7395\widehat{q} - 0.0857\delta + 0.7685(q - \widehat{q}) \;.
\end{aligned}$$

The observer produces estimates of the states and these are used in the control law we established previously (with poles at -0.3),

$$u = 0.8451\widehat{\theta} + 1.4733\widehat{w} - 0.9807\widehat{q} \;.$$

The system is subjected to an initial disturbance $\theta = 30$ degrees, while for the observer we use $\widehat{\theta} = 0$ since the observer does not know the true value of θ. The results of the simulation are presented in Figure 18 where it can be seen that θ approaches zero in much the same way as for the complete state measurement case of the previous section. The estimate $\widehat{\theta}$ approaches the true value of θ quickly.

If we were to reduce the absolute value of the observer poles, say to -0.1, -0.11, -0.12 we are faced with the following pathological situation: In order for the control law to return the system to its equilibrium, it needs an accurate estimate of the states as quickly as possible. Since the observer poles, however, are less negative than the controller poles this estimate

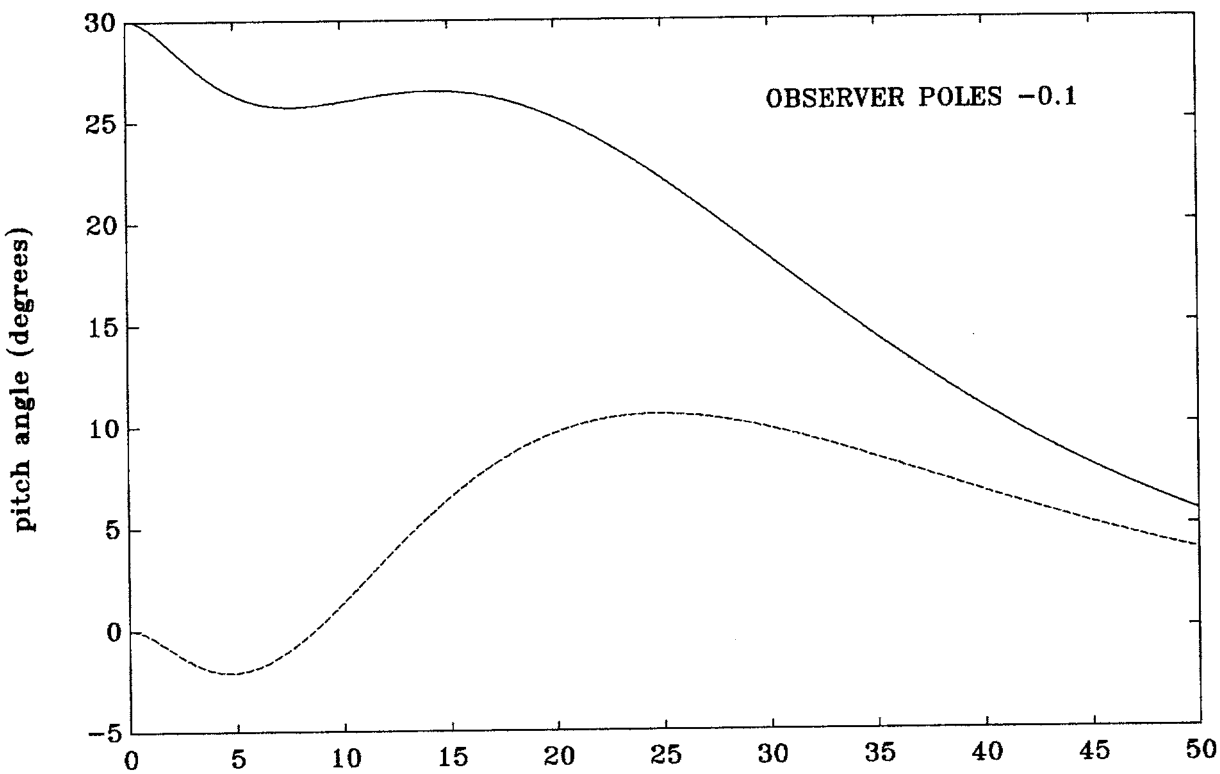

Figure 3: Compensator response for slow observer poles

will be slow which means that it will take longer for the control law to stabilize the system to its equilibrium point. Indeed, in such a case the observer gains are

$$L = \begin{bmatrix} 0.8664 \\ -0.4817 \\ -0.7315 \end{bmatrix} ,$$

and the results of the simulation are shown in Figure 19. It can be seen that the response of the system is slow; even though the control poles were specified at -0.3 the response looks more like the -0.1 controller poles of the perfect state knowledge case of the previous section (why?).

It appears that we need to have the observer poles as negative as possible, compared to the closed loop control poles. A good rule–of–thumb practice is twice as negative. Beyond that we do not gain much and we run into problems with sensor noise, more about this later. For now, it is enough to recognize the fact that as the observer poles become more negative, the elements of L become larger in absolute value (verify this using MATLAB) and any kind of sensor noise that gets into our measurements will be magnified. There is a limit on how large the elements of L can be and this depends on the quality of our sensors. This is the optimal observer design or Kalman filter problem which we discuss later.

3.5 Reduced Order Observers

The previously developed observer is usually called a full order observer: its order is the same as that of the system. A full order observer estimates all the states in a system, regardless whether they are measured or not. This does not seem to be too bad, except imagine we have a system with ten states and we can measure eight of them; wouldn't it be better to estimate two instead of all ten states? The formalization of this procedure leads to the reduced order estimator.

Suppose we can measure some of the state variables contained in x. We partition the state vector x into two sets,

$$\begin{aligned} x_1 &: \text{ variables that can be measured directly ,} \\ x_2 &: \text{ variables that cannot be measured directly .} \end{aligned}$$

The state equations are broken down to

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u , \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u , \end{aligned}$$

and the observation equation is

$$y = C_1 x_1 ,$$

where C_1 is square and nonsingular matrix. The full order observer for the states is then

$$\begin{aligned} \dot{\widehat{x}}_1 &= A_{11}\widehat{x}_1 + A_{12}\widehat{x}_2 + B_1 u + L_1(y - C_1\widehat{x}_1) , \\ \dot{\widehat{x}}_2 &= A_{21}\widehat{x}_1 + A_{22}\widehat{x}_2 + B_2 u + L_2(y - C_1\widehat{x}_1) . \end{aligned}$$

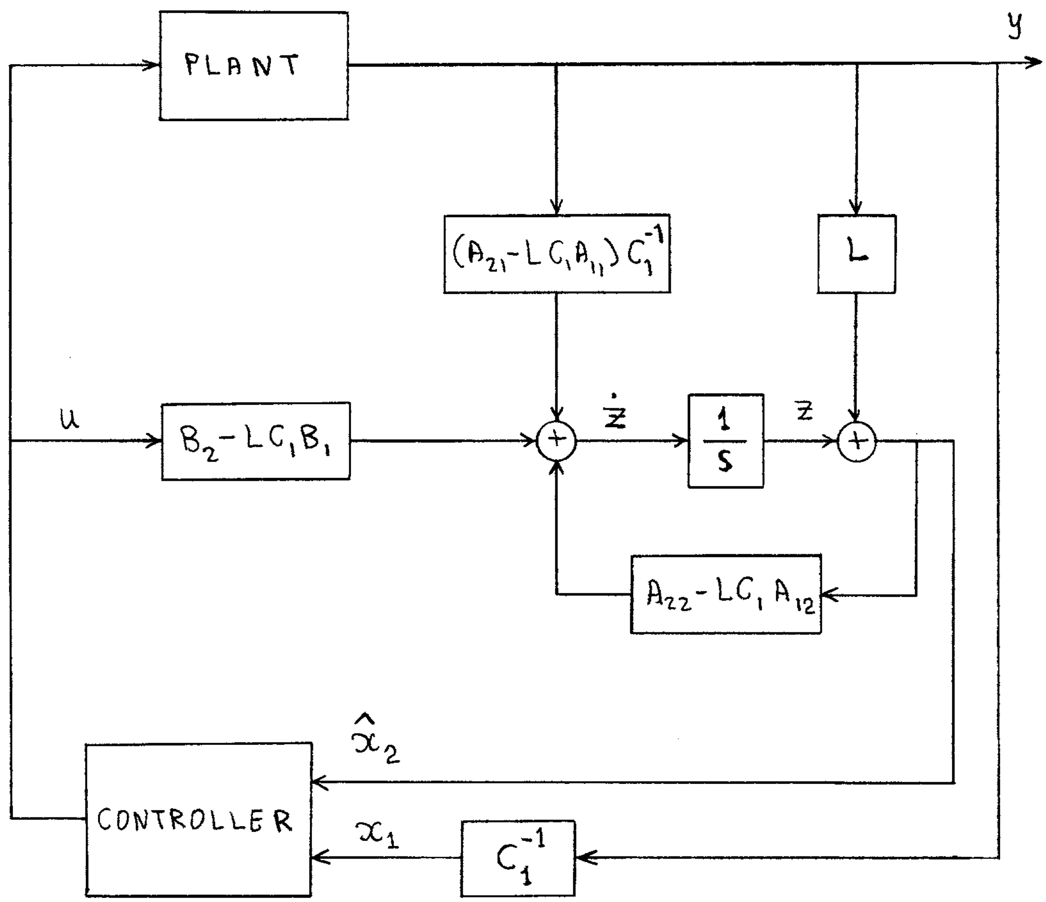

Figure 4: Block diagram of reduced order observer

But why take the trouble to implement the first observer equation for $\widehat{x}_1$ when we can solve for x_1 directly?

$$\widehat{x}_1 = x_1 = C_1^{-1} y \ .$$

In this case the observer for those states that cannot be measured directly becomes

$$\dot{\widehat{x}}_2 = A_{21} C_1^{-1} y + A_{22} \widehat{x}_2 + B_2 u \ ,$$

which is a dynamic system of the same order as the number of state variables that cannot be measured directly. The dynamic behavior of this reduced order observer is governed by the eigenvalues of A_{22}, a matrix over which the designer has no control. Since there is no assurance that the eigenvalues of A_{22} are suitable, we need a more general system for the reconstruction of $\widehat{x}_2$. We take

$$\widehat{x}_2 = Ly + z \ ,$$

where

$$\dot{z} = Fz + Gy + Hu \ .$$

Define the estimation error

$$e = x - \widehat{x} = \begin{bmatrix} x_1 - \widehat{x}_1 \\ x_2 - \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ e_2 \end{bmatrix} \ ,$$

and we get

$$\begin{aligned} \dot{e}_2 &= \dot{x}_2 - \dot{\widehat{x}}_2 \\ &= A_{21} x_1 + A_{22} x_2 + B_2 u - L\dot{y} - \dot{z} \\ &= A_{21} x_1 + A_{22} x_2 + B_2 u - LC_1 \dot{x}_1 - Fz - Gy - Hu \\ &= A_{21} x_1 + A_{22} x_2 + B_2 u - LC_1 (A_{11} x_1 + A_{12} x_2 + B_1 u) \\ &\qquad - F(\widehat{x}_2 - Ly) - Gy - Hu \ . \end{aligned}$$

Since

$$\widehat{x}_2 - Ly = x_2 - e_2 - Ly = x_2 - e_2 - LC_1 x_1 \ ,$$

we get

$$\begin{aligned} \dot{e}_2 &= Fe_2 + (A_{21} - LC_1 A_{11} - GC_1 + FLC_1) x_1 \\ &\qquad + (A_{22} - LC_1 A_{12} - F) x_2 + (B_2 - LC_1 B_1 - H) u \ . \end{aligned}$$

In order for the error to be independent of x_1, x_2, and u, the matrices multiplying x_1, x_2, and u must vanish

$$\begin{aligned} F &= A_{22} - LC_1 A_{12} \ , \\ H &= B_2 - LC_1 B_1 \ , \\ G &= (A_{21} - LC_1 A_{11}) C_1^{-1} + FL \ . \end{aligned}$$

Then

$$\dot{e}_2 = Fe_2 \ ,$$

and for stability the eigenvalues of F must lie in the left half s–plane. Therefore, we see that the problem of reduced order observer is similar to the full order observer with $(A_{22}-LC_1A_{12})$ playing the role of $(A-LC)$. The block diagram schematic appears as shown in Figure 20.

Example: Consider the submarine problem, and assume that both the pitch angle θ and pitch rate q are available through measurements. What we need is to estimate the vertical translational (heave) velocity w. Let's design a reduced order observer to do the job. We start with our equations of motion and we re–write them so that the variables that are measurable go first

$$\begin{aligned}\dot\theta &= q\ ,\\ \dot q &= a_{21}Uw + a_{22}Uq + a_{23}z_{GB}\theta + b_2U^2\delta\ ,\\ \dot w &= a_{11}Uw + a_{12}Uq + a_{13}z_{GB}\theta + b_1U^2\delta\ .\end{aligned}$$

In matrix form we have

$$\begin{bmatrix}\dot\theta\\ \dot q\\ \dot w\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ a_{23}z_{GB} & a_{22}U & a_{21}U\\ a_{13}z_{GB} & a_{12}U & a_{11}U\end{bmatrix}\begin{bmatrix}\theta\\ q\\ w\end{bmatrix} + \begin{bmatrix}0\\ b_2U^2\\ b_1U^2\end{bmatrix}\delta\ ,$$

and the measurement equation is

$$y = \begin{bmatrix}\theta\\ q\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta\\ q\\ w\end{bmatrix}\ .$$

Therefore, the matrices are

$$\begin{aligned}&x_1 = \begin{bmatrix}\theta\\ q\end{bmatrix},\ x_2 = \begin{bmatrix} w\end{bmatrix},\\ &A_{11} = \begin{bmatrix}0 & 1\\ a_{23}z_{GB} & a_{22}U\end{bmatrix},\ A_{12} = \begin{bmatrix}0\\ a_{21}U\end{bmatrix},\ A_{21} = \begin{bmatrix}a_{13}z_{GB} & a_{12}U\end{bmatrix},\ A_{22} = \begin{bmatrix}a_{11}U\end{bmatrix},\\ &B_1 = \begin{bmatrix}0\\ b_2U^2\end{bmatrix},\ B_2 = \begin{bmatrix}b_1U^2\end{bmatrix},\ C_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ L = \begin{bmatrix}\ell_1 & \ell_2\end{bmatrix}.\end{aligned}$$

The reduced order observer equations are

$$\begin{aligned}\widehat w &= \ell_1\theta + \ell_2 q + z\ ,\\ \dot z &= Fz + Gy + H\delta\ .\end{aligned}$$

Following the design procedure we have

$$F = a_{11}U - \begin{bmatrix}\ell_1 & \ell_2\end{bmatrix}\begin{bmatrix}0\\ a_{21}U\end{bmatrix} = a_{11}U - \ell_2 a_{21}U = p\ ,$$

where p is the desired observer pole (F here is a scalar since there is only one state variable to be estimated). We see that ℓ_1 plays no role in determining F and, therefore, we can choose

$$\ell_1 = 0\ ,$$

for simplicity. The other observer gain ℓ_2 is computed from

$$\ell_2 = \frac{a_{11}U - p}{a_{21}U} \ .$$

Then we get

$$\begin{aligned}
H &= b_1U^2 - \begin{bmatrix} 0 & \ell_2 \end{bmatrix} \begin{bmatrix} 0 \\ b_2U^2 \end{bmatrix} \\
&= b_1U^2 - \ell_2 b_2 U^2 \ , \\
G &= A_{21} - LA_{11} + FL \\
&= \begin{bmatrix} a_{13}z_{GB} & a_{12}U \end{bmatrix} - \begin{bmatrix} 0 & \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ a_{23}z_{GB} & a_{22}U \end{bmatrix} + p \begin{bmatrix} 0 & \ell_2 \end{bmatrix} \\
&= \begin{bmatrix} a_{13}z_{GB} & a_{12}U \end{bmatrix} - \begin{bmatrix} \ell_2 a_{23}z_{GB} & \ell_2 a_{22}U \end{bmatrix} + \begin{bmatrix} 0 & p\ell_2 \end{bmatrix} \\
&= \begin{bmatrix} a_{13}z_{GB} - \ell_2 a_{23}z_{GB} & a_{12}U - \ell_2 a_{22}U + p\ell_2 \end{bmatrix} \ .
\end{aligned}$$

The observer equations are then

$$\begin{aligned}
\dot{z} &= pz + (a_{13}z_{GB} - \ell_2 a_{23}z_{GB})\theta + (a_{12}U - \ell_2 a_{22}U + p\ell_2)q \\
&\qquad + (b_1U^2 - \ell_2 b_2 U^2)\delta \ , \\
\widehat{w} &= \ell_2 q + z \ .
\end{aligned}$$

Simulation results for control poles at -0.3 and observer pole at -0.6 are shown in Figure 21, in terms of w and $\widehat{w}$ versus time. In this simulation the initial conditions were changed to $\theta = q = 0$, $w = 0.5$ ft/sec, and $\widehat{w} = 0$. This was done to better show the convergence of $\widehat{w}$ to the true value w. The same remarks concerning selection of observer poles apply for the reduced order observer as for the full order observer design.

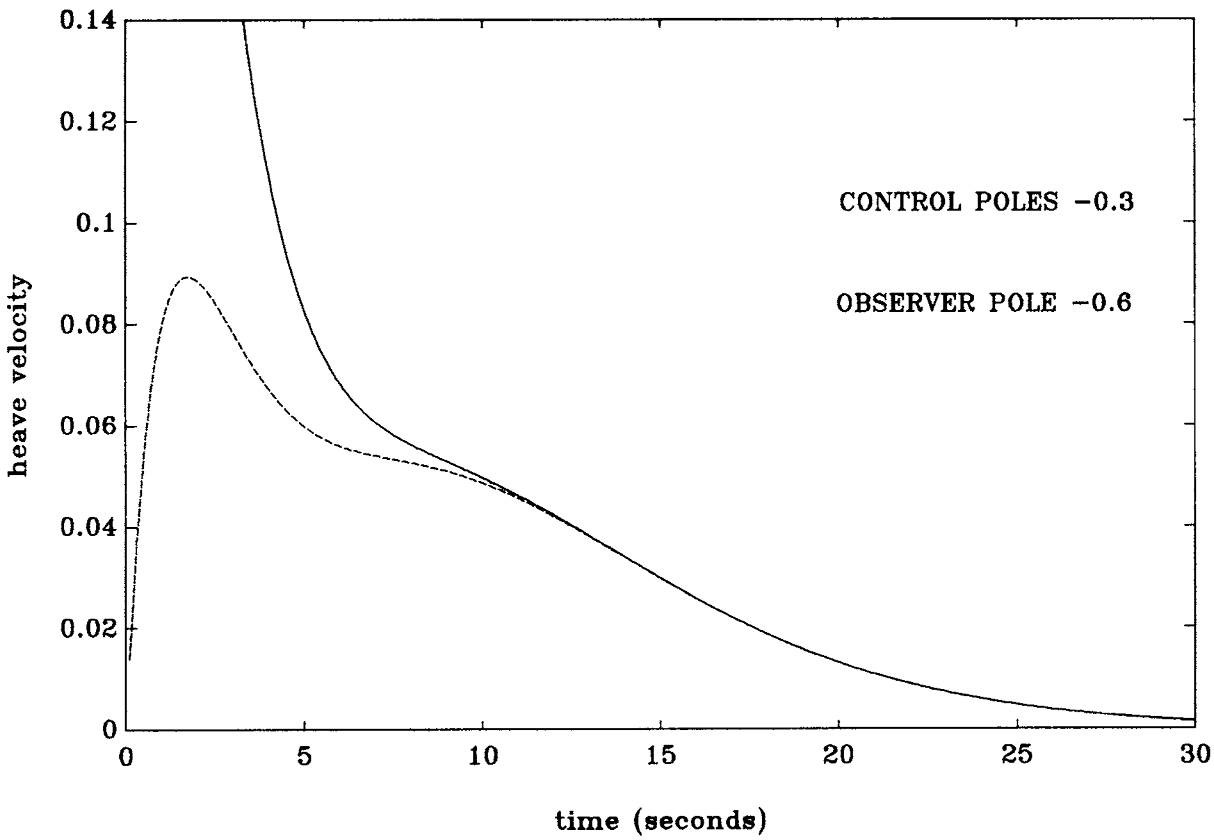

Figure 5: Response of the reduced order estimator