2 CONTROLLER DESIGN

The control design problem can be stated as follows: Given the system

$$\dot{x} = \underbrace{A}_{n \times n} \underbrace{x}_{n \times 1} + \underbrace{B}_{n \times 1} \underbrace{u}_{1 \times 1} ,$$

how do we find u such that x behaves nicely? We consider for now single input systems (u is scalar and B is a vector), the multiple input case is studied later. We are particularly interested in closed loop control, where u is a function of the state x. The case where u is an explicit function of time only and not x is called open loop control and is studied under system dynamics. Since we are using the state x to determine the control effort $u(x)$ we call it *feedback* control.

2.1 Pole Placement

The simplest case of feedback control $u(x)$ is when u is linear in x,

$$u = -\underbrace{K}_{1 \times n} x ,$$

where K is the feedback gain vector to be determined. Substituting $u = -Kx$ into $\dot{x} = Ax + Bu$ we get

$$\begin{aligned} \dot{x} &= Ax - BKx , \quad \text{or} \\ \dot{x} &= (A - BK)x . \end{aligned}$$

The *actual* characteristic equation of this closed loop system is given by

$$\det[A - BK - sI] = 0 .$$

We can now pick K such that the actual characteristic equation assumes any desired set of eigenvalues. If we choose the desired locations of the closed loop poles at $s = s_i$ for $i = 1, \ldots, n$, the *desired* characteristic equation is

$$(s - s_1)(s - s_2) \ldots (s - s_n) = 0 .$$

The required values of K are obtained then by matching coefficients in the two polynomials of the actual and desired characteristic equations.

Consider the example:

$$A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} , \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

The open loop eigenvalues are

$$\det[sI - A] = \left| \begin{matrix} s-1 & 5 \\ 5 & s-1 \end{matrix} \right] = 0 \Longrightarrow (s-1)^2 - 5 = 0 \Longrightarrow (s-6)(s+4) = 0 ,$$

so we have an unstable system with no control. If the pair (A, B) is controllable we are guaranteed that we can pick the elements of K to produce an arbitrary characteristic equation. In this case we have

$$AB = \begin{bmatrix} 1 \\ -5 \end{bmatrix} , \quad \mathcal{C} = \begin{bmatrix} 1 & 1 \\ 0 & -5 \end{bmatrix} , \quad \det \mathcal{C} = -5 \neq 0 ,$$

so the system is controllable. Now suppose we want closed loop eigenvalues at $-10 \pm 10i$ so that we get a damping ratio $\zeta = 0.707$. The desired closed loop characteristic equation is

$$(s + 10 - 10i)(s + 10 + 10i) = s^2 + 20s + 200 = 0 .$$

Form the matrix

$$A - BK = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 - k_1 & -5 - k_2 \\ -5 & 1 \end{bmatrix} ,$$

and the actual closed loop characteristic equation is

$$\begin{aligned} &\det[A - BK - sI] = \begin{vmatrix} 1 - k_1 - s & -5 - k_2 \\ -5 & 1 - s \end{vmatrix} = 0 , \quad \text{or} \\ &1 - k_1 - s - s + k_1 s + s^2 - 25 - 5k_2 = 0 , \quad \text{or} \\ &s^2 + (k_1 - 2)s + (-k_1 - 5k_2 - 24) = 0 , \end{aligned}$$

requiring

$$\begin{aligned} -2 - k_1 &= 20 , \\ -k_1 - 5k_2 - 24 &= 200 . \end{aligned}$$

Solving this we get

$$\begin{aligned} k_1 &= 22 , \\ k_2 &= -\frac{246}{5} , \end{aligned}$$

and the control law is

$$u = -k_1 x_1 - k_2 x_2 = -22x_1 + \frac{246}{5} x_2 .$$

Note that these gains may be impossible or impractical to build for this system. This would require some compromise in the specification which led to the desired closed loop eigenvalues. In general, the above approach yields a system of n linear equations to be solved for the n elements of K provided (A, B) is controllable. This method is known as *pole placement.*

Example: Consider the submarine equations

$$\begin{aligned} \dot{\theta} &= q , \\ \dot{w} &= a_{13} z_{GB} \theta + a_{11} U w + a_{12} U q + b_1 U^2 \delta , \\ \dot{q} &= a_{23} z_{GB} \theta + a_{21} U w + a_{22} U q + b_2 U^2 \delta \end{aligned}$$

Let the control law be

$$\delta = -k_1\theta - k_2 w - k_3 q \ .$$

Substituting into the equations we get the closed loop system

$$\begin{aligned}
\dot{\theta} &= q \ , \\
\dot{w} &= (a_{13}z_{GB} - b_1 U^2 k_1)\theta + (a_{11}U - b_1 U^2 k_2)w + (a_{12}U - b_1 U^2 k_3)q \ , \\
\dot{q} &= (a_{23}z_{GB} - b_2 U^2 k_1)\theta + (a_{21}U - b_2 U^2 k_2)w + (a_{22}U - b_2 U^2 k_3)q \ ,
\end{aligned}$$

or, in matrix form,

$$\begin{bmatrix} \dot{\theta} \\ \dot{w} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a_{13}z_{GB} - b_1 U^2 k_1 & a_{11}U - b_1 U^2 k_2 & a_{12}U - b_1 U^2 k_3 \\ a_{23}z_{GB} - b_2 U^2 k_1 & a_{21}U - b_2 U^2 k_2 & a_{22}U - b_2 U^2 k_3 \end{bmatrix} \ .$$

The characteristic equation of the closed loop system is

$$\det \begin{vmatrix} 0 - s & 0 & 1 \\ a_{13}z_{GB} - b_1 U^2 k_1 & a_{11}U - b_1 U^2 k_2 - s & a_{12}U - b_1 U^2 k_3 \\ a_{23}z_{GB} - b_2 U^2 k_1 & a_{21}U - b_2 U^2 k_2 & a_{22}U - b_2 U^2 k_3 - s \end{vmatrix} = 0 \ ,$$

and after some algebra this reduces to

$$\begin{aligned}
& s^3 + (-D_1' + A_2 k_2 + A_3 k_3)s^2 + (-B_1 k_1 - B_2 k_2 - B_3 k_3 - D_2')s \\
& \quad +(-C_1 k_1 - C_2 k_2 - D_3') = 0 \ ,
\end{aligned}$$

where we have denoted

$$\begin{aligned}
& A_2 = b_1 U^2 \ , \quad A_3 = -B_1 = b_2 U^2 \ , \\
& B_2 = (b_1 a_{22} - b_2 a_{12})U^3 \ , \quad B_3 = C_1 = (b_2 a_{11} - b_1 a_{21})U^3 \ , \\
& C_2 = (a_{23}b_1 - a_{13}b_2)U^2 z_{GB} \ , \quad D_1' = (a_{11} + a_{22})U \ , \\
& D_2' = a_{23}z_{GB} + (a_{12}a_{21} - a_{11}a_{22})U^2 \ , \quad D_3' = (a_{13}a_{21} - a_{11}a_{23})z_{GB}U \ .
\end{aligned}$$

Now assume that the we wish to place the closed loop poles at $-p_1$, $-p_2$, $-p_3$. This means that the desired characteristic equation is

$$\begin{aligned}
& (s + p_1)(s + p_2)(s + p_3) = 0 \ , \quad \text{or} \\
& s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0 \ ,
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1 &= p_1 + p_2 + p_3 \ , \\
\alpha_2 &= p_1 p_2 + p_2 p_3 + p_3 p_1 \ , \\
\alpha_3 &= p_1 p_2 p_3 \ .
\end{aligned}$$

Then, the control gains can be computed by equating coefficients of the actual and the desired characteristic equations

$$\begin{aligned}
& A_2 k_2 + A_3 k_3 = -\alpha_1 - D_1' \ , \\
& B_1 k_1 + B_2 k_2 + B_3 k_3 = \alpha_2 + D_2' \ , \\
& C_1 k_1 + C_2 k_2 = \alpha_3 + D_3' \ .
\end{aligned}$$

This method of equating coefficients is feasible only for small systems and it always produces a linear system in the unknown gains k_i.

The above approach can be simplified if the system is written in its control canonical form

$$\dot{x}' = A'x' + B'u \ , \quad y = C'x' \ ,$$

and we are seeking a control law of the form

$$u = -K'x' \ .$$

As an example say the open loop characteristic equation is

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \ ,$$

and the state space form of the system is

$$\begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \\ \dot{x}'_3 \\ \dot{x}'_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \ ,$$

and

$$y = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix} \ ,$$

with the control law

$$u = -\begin{bmatrix} k'_1 & k'_2 & k'_3 & k'_4 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix} \ .$$

The transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \ .$$

Observe that no algebra is needed here, if we have the transfer function we can write the control canonical form directly.

We can select now our desired closed loop characteristic equation

$$s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 \ .$$

Then

$$A - BK' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 - k'_1 & -a_1 - k'_2 & -a_2 - k'_3 & -a_3 - k'_4 \end{bmatrix} \ ,$$

with closed loop characteristic equation

$$s^4 + (a_3 + k'_4)s^3 + (a_2 + k'_3)s^2 + (a_1 + k'_2)s + (a_0 + k'_1) = 0 \ .$$

Again, since we are working with a control canonical form, no algebra has been necessary so far. We can now solve for the gains directly without solving a system of linear equations

$$\begin{aligned} k'_1 &= -a_0 + \alpha_0 \ , \\ k'_2 &= -a_1 + \alpha_1 \ , \\ k'_3 &= -a_2 + \alpha_2 \ , \\ k'_4 &= -a_3 + \alpha_3 \ , \end{aligned}$$

and the control law is

$$\begin{aligned} u &= -k'_1 x'_1 - k'_2 x'_2 - k'_3 x'_3 - k'_4 x'_4 \ , \\ &= -(-a_0 + \alpha_0)x'_1 - (-a_1 + \alpha_1)x'_2 - (-a_2 + \alpha_2)x'_3 - (-a_3 + \alpha_3)x'_4 \ . \end{aligned}$$

Draw a block diagram of the system before and after feedback control; do you see what happens?

To summarize, if we have a system

$$\dot{x}' = Ax' + Bu \ ,$$

in the control canonical form, we can introduce a feedback control law

$$u = -K'x' \ ,$$

with feedback gains

$$K' = -a + \alpha \ ,$$

where

$$\begin{aligned} a &= \text{coefficients of original characteristic equation} \ , \\ \alpha &= \text{coefficients of desired characteristic equation} \ . \end{aligned}$$

If the system is not in the control canonical form we have to transform it. Suppose that the original state x is transformed into x' through the transformation

$$x' = Tx \ ,$$

and

$$\dot{x} = Ax + Bu \ ,$$

becomes

$$\dot{x}' = TAT^{-1}x' + TBu \ .$$

For the transformed system, which is in the control canonical form,

$$u = -K'x' \ ,$$

where

$$K' = -a' + \alpha = -a + \alpha \ ,$$

since the characteristic equation is invariant under a change of state variables. The control law is

$$\begin{aligned} u &= -K'x' \ , \\ &= -K'Tx \ , \\ &= -Kx \ , \end{aligned}$$

where

$$\underbrace{K}_{1\times n} = \underbrace{K'}_{1\times n} \underbrace{T}_{n\times n} \ ,$$

is the gain in the original system. This can also be written as

$$\underbrace{K^T}_{n\times 1} = \underbrace{T^T}_{\text{transpose}} \underbrace{(-a+\alpha)}_{n\times n} \ .$$

We only need to find the transformation matrix T which will transform any system into its control canonical form. The desired matrix T is the product of two matrices

$$T = VU \ ,$$

where U is the inverse of the controllability matrix $\mathcal{C}$

$$U = \mathcal{C}^{-1} \ .$$

Notice that if the system is uncontrollable, U does not exist. Matrix V is given by

$$V = W^{-1} \ ,$$

where

$$W = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdot & \cdot & \cdot & a_1 \\ 0 & 1 & a_{n-1} & \cdot & \cdot & \cdot & a_2 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & a_3 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} ;$$

the first row is formed by the coefficients of the characteristic polynomial of A

$$\det[A - sI] = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \ ,$$

and the other rows are pushed left by one at a time. Therefore, the desired control law is

$$K^T = \left[(\mathcal{C}W)^T\right]^{-1} (-a + \alpha) \ .$$

Now that we have a formula for the gains of a controllable single input system that will place the poles at any desired location, several questions arise:

1. If the closed loop poles can be placed anywhere, where should they be placed?

2. How can the technique be extended to multiple input systems?

3. What if not all states are available for feedback and we have to use output measurements only?

4. What do we do if we have external disturbances and we want to track a reference input?

5. How do we handle effects of sensor noise?

6. Can we optimize the performance of a control system?

The above questions are the subject of the remaining of these notes.

2.2 Pole Location Selection

For a second order system we may have some transient response specifications, such as rise time, percent overshoot, or settling time. These result in an allowable region in the s–plane from which we can easily get the desired locations of the poles. For higher order systems we can employ the concept of dominant roots, select two roots as dominant which means that we want to place the remaining roots more negative so that the transient response is not affected significantly. In selecting poles for a physical system we need to look at the physics; we can not specify poles that are too negative, for example. This would demand a very small time constant for the control system and the physical system may not be able to react that fast.

The control law $u = -Kx$ implies that for a given state x the larger the gain, the larger the control input. In practice, however, there are limits on u: actuator size and saturation. Occasional control saturation is not serious and may be even desirable; a system which never saturates is probably overdesigned.

Example: Control design by pole placement is very easy using MATLAB, the appropriate command is `place` which accepts as inputs the A, B matrices and a vector of the desired closed loop poles, and returns the gain vector K. For example, consider the submarine equations

$$\begin{bmatrix} \dot{\theta} \\ \dot{w} \\ \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0.0135 & -0.3220 & -0.7102 \\ -0.0360 & 0.1260 & -0.7395 \end{bmatrix}}_{A} \begin{bmatrix} \theta \\ w \\ q \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0.0322 \\ -0.0857 \end{bmatrix}}_{B} \delta \, .$$

Say we want to design a control law to stabilize the submarine to a level flight path at $\theta = 0$. We want to be able to return to level after an initial small disturbance in θ within the time it takes to travel one ship length, this is reasonable. Since the boat is about 17 feet long and it travels at 5 ft/sec, that time is about 3.5 seconds; so we want the control law to have a time constant of 3 seconds. This means we want to place the closed loop poles at approximately -0.3. Using `place` we specify poles at -0.3, -0.31, -0.32 (`place` does not like poles that are exactly the same) and we find the gains in the control law

$$\delta = -(-0.8451\theta - 1.4733w + 0.9807q) \; .$$

Using a simulation program we plot the response starting from 30 degrees positive (bow up) pitch angle. We also set a limit in the dive plane angle between ± 0.4 radians. We can see from the results that initially the planes saturate at the upper limit and they come off as θ approaches zero. For comparison, we show the response with no control (planes fixed at zero). If we specify more negative poles, at -0.9, -0.91, -0.92, the control law becomes

$$\delta = -(-31.61147\theta - 1.2581w - 24.6634q) \; .$$

Observe how unrealistically high these gains are: for a unit change in the pitch angle θ our controller demands 32 degrees of plane action! The response is also shown in the figure; there is more plane activity than in the previous case. However, since we hit the saturation limit, the response is not any faster and it overshoots the desired value. If we specify less negative poles at -0.1, -0.11, -0.12, we end up with a control law

$$\delta = -(0.3640\theta - 1.2581w + 8.0657q) \; .$$

This is a very soft control law, it takes considerably longer for θ to reach zero and there is very limited plane activity.

From the above results, that are plotted in Figures 14 and 15, we can see that:

- Poles that are specified too negative will not necessarily result in faster response for a physical system; we may reach the hardware limitations of the system.
- Poles that are specified too negative will result in a high gain tight control law which will exhibit continuous control action; the system will over–respond to everything, including measurement noise.
- Poles that are specified not negative enough will result in soft response with a very quite control system that hardly works at all.
- Proper pole selection can be achieved by knowing the physics of the system we are trying to control, and by a trial–and–error simulation process.

The effect of control system gain on pole locations can be appreciated by considering the formula

$$K^T = \left[(\mathcal{C}W)^T \right]^{-1} (-a + \alpha) \; .$$

The gains are proportional to the amounts that the poles are to be moved: the less the poles are moved the smaller the gain matrix and therefore the control effort. It is also seen that the control system gains are inversely proportional to the controllability test matrix $\mathcal{C}$. The

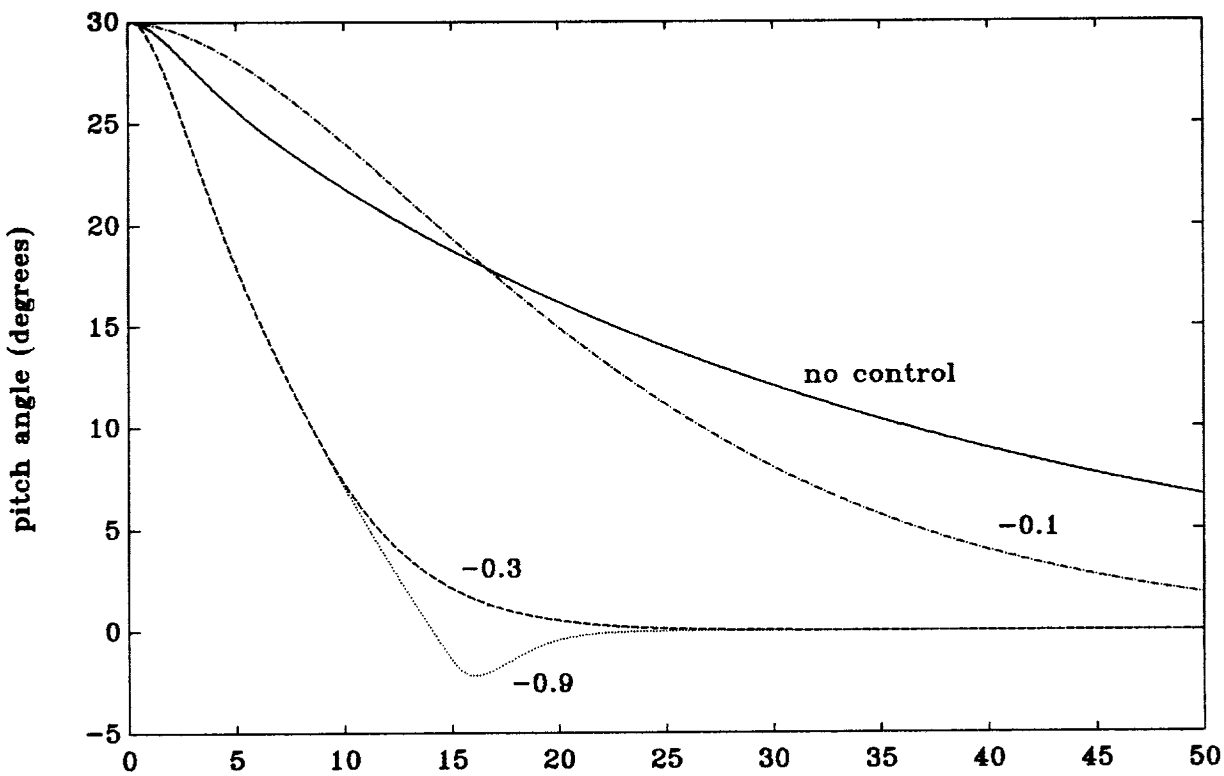

Figure 1: Pitch angle versus time for different closed loop poles

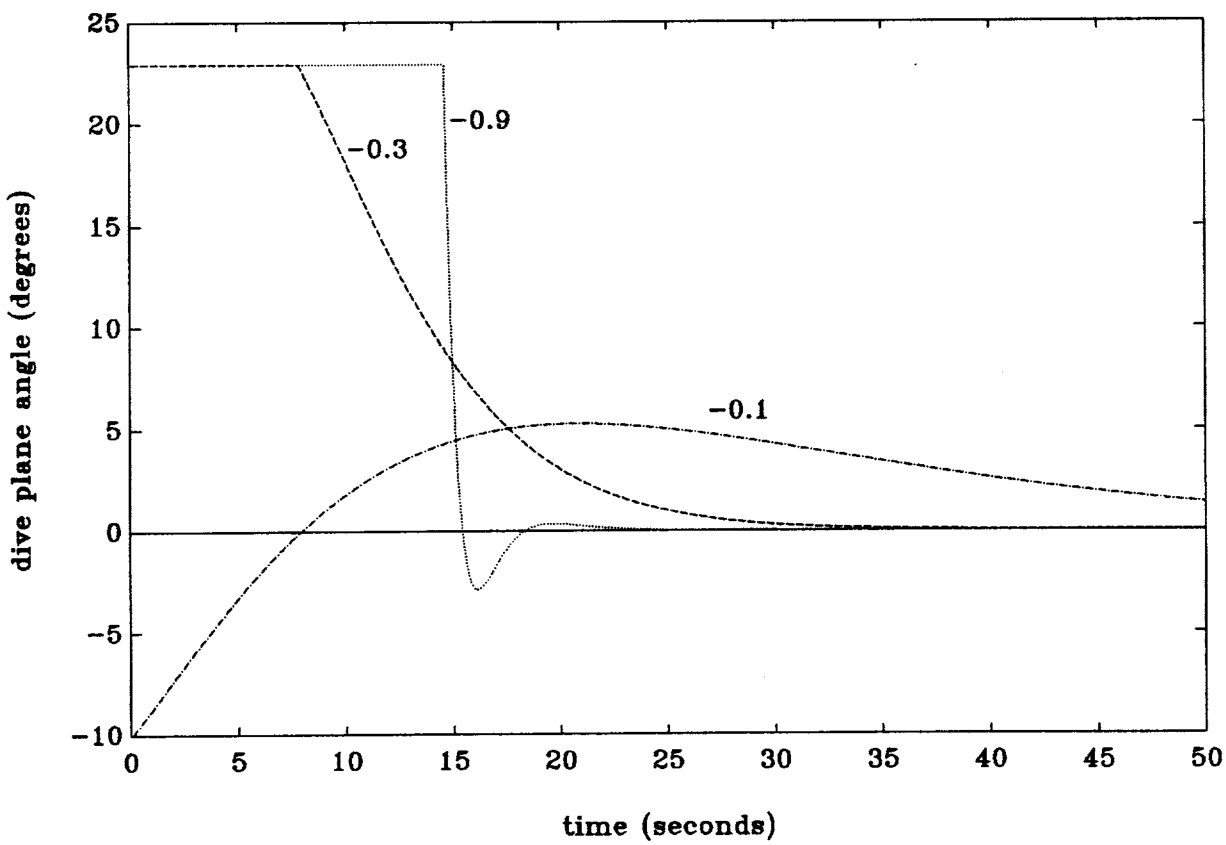

Figure 2: Control effort for different closed loop poles

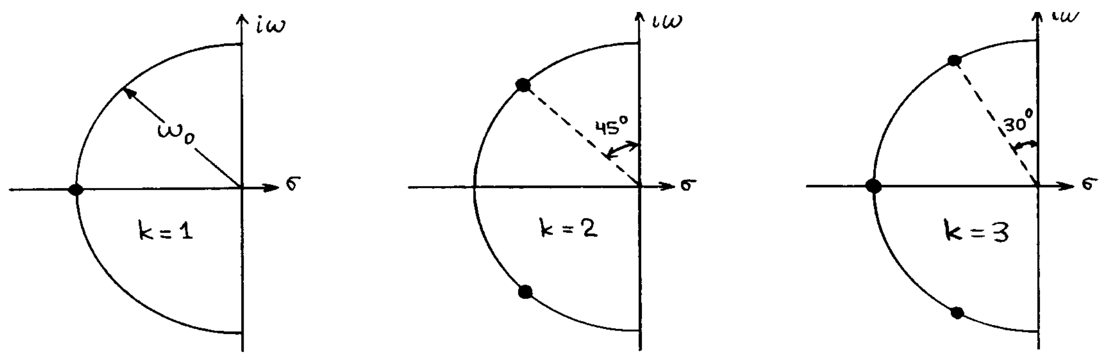

Figure 3: Butterworth pole configurations

less controllable the system, the larger the gains that are needed to make a change in the system poles.

Some broad guidelines for pole selection are:

- Select a bandwidth high enough to achieve desired speed of response.
- Keep the bandwidth low enough to avoid exciting unmodeled high frequency effects and undesired response to noise.
- Place the poles at approximately uniform distances from the origin for efficient use of the control effort.

We can also use standard characteristic polynomials such as minimizing the ITAE criterion, Bessel transfer functions, or Butterworth pole configurations. A typical sketch of the Butterworth poles is shown in Figure 16.

The closed loop poles tend to radiate out from the origin along the spokes of a wheel in the left half plane as given by the roots of

$$\left(\frac{s}{\omega_0}\right)^{2k} = (-1)^{k+1} ,$$

where k is the number of roots in the left half plane and ω_0 the natural frequency. In the absence of any other consideration, a Butterworth configuration is often suitable. Note, however, that as the order of the system k becomes high, one pair of poles comes very close to the imaginary axis. It might be desirable then to move these poles further into the left half plane.

Optimal control strategies can also be used to optimize some performance index. One common choice here is

$$\min\ J = \int_0^T (x^T Q x + u^T R u)\, dt ,$$

where

$$\begin{aligned} Q &= \text{weighting matrix of the error } x , \\ R &= \text{weighting matrix of the control effort } u . \end{aligned}$$

This is the Linear Quadratic Regulator problem which is studied later in these notes.

2.3 Multiple Input Systems

If the dynamic system under consideration

$$\dot{x} = Ax + Bu ,$$

has more than one inputs, that is B has more than one columns, then the gain matrix K in the control law

$$u = -Kx ,$$

has more than one rows. Since each row of K furnishes n adjustable gains, it is clear than in a controllable system there will be more gains available than needed to place all of the closed loop poles. If we have m inputs, then the equation

$$\det |A - BK| = \text{ specified characteristic polynomial}$$

gives n equations with $n \times m$ unknowns. More than one solutions exist in general. This gives the designer more flexibility: it is possible to specify all the closed loop poles and still be able to satisfy other requirements. There are several possibilities here, some of them are briefly discussed below.

1. We can make one control proportional (or related) to the other. For example if we have a two input system

$$\dot{x} = Ax + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ,$$

we can choose

$$u_2 = \lambda u_1 ,$$

with λ some selected constant of proportionality, and the system becomes

$$\dot{x} = Ax + \begin{bmatrix} b_1 + \lambda b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ,$$

which is single input. The underlying physics should be the guidance for selection in this method. For example, say that our submarine is equipped with two inputs for depth control: independent stern and bow planes, call them δ_s and δ_b. If rapid depth change is what we want at a regular cruising speed then it makes sense to assume that $\delta_b = -\delta_s$. This deflects the bow planes differentially than stern planes and produces maximum control authority through maximizing the vehicle pitching moment. If on the other hand, the vehicle is equipped with vertical stern and bow thrusters and is operating near hover, it is natural to command the same instead of opposite values for the two control inputs in order to achieve depth control.

2. Another possible method of selecting a particular structure for the gain matrix is to make each control variable depend on a different group of state variables that are physically more closely related to that control variable than to the other control variables. For example, suppose that our submarine is equipped with stern planes and sail planes at about amidships. Then is makes sense to use the stern planes to directly control pitch angle and the sail planes for direct depth control. Formally, what we are doing in this case is to specify not just the eigenvalues of the closed loop matrix but also (some of) its eigenvectors. This achieves a more precise shaping of the response.

3. Another possibility might be to set some of the gains to zero. For example, it is possible (sometimes) to place the closed loop poles at the desired locations with a gain matrix which has a column of zeros. This means that the state variable corresponding to that column is not needed in the generation of any of the control signals in the vector u, and hence there is no need to measure (or estimate) that state variable. This simplifies the resulting control system structure. If all the state variables, except those corresponding to columns of zeros in the gain matrix, are accessible for measurement then there is no need for an observer to estimate the state variables that cannot be measured. A very simple and robust control system is the result.

Hand calculation of the system of equations to be solved for the gains is possible for the multiple input case just like the single input. The only difference here is that unlike the single input where we always end up with a system of linear simultaneous equations in k_i, for multiple inputs it is possible to come up with a nonlinear system for k_{ij}.