1 INTRODUCTION

Unlike "classical control" theory (ME 3801) which is based on Laplace transform representations, "modern control" deals directly with systems described in ordinary differential equation form. We assume that given a physical system, we have already developed our equations of motion, in other words the modeling part is complete. The goal here is to affect the dynamic response of the system such that it performs a specific task in a satisfactory way. The first thing we have to do is to rewrite our differential equations of motion in their *state space* form.

1.1 State Variable System Description

The *state* is a set of quantities such that given initial conditions $x(t_0)$ and all future inputs $u(t)$, all future response $x(t)$ for $t > t_0$ is uniquely determined. If not enough initial conditions are specified, then more than one responses may be obtained; if too many initial conditions are specified, then a solution may not be possible. Therefore, we can see that for any dynamical system the number of states is unique; the choice, however, is not.

The *state equations* are a coupled set of first–order linear differential equations in the state variables; i.e.,

$$\dot{x} = Ax + Bu \ ,$$

where

$$\begin{aligned} x &: \ \text{state vector, } n \times 1 \ , \\ A &: \ \text{open–loop dynamics matrix, } n \times n \ , \\ u &: \ \text{control vector, } m \times 1 \ , \\ B &: \ \text{control distribution matrix, } n \times m \ , \end{aligned}$$

along with the output equation

$$y = Cx \ ,$$

where

$$\begin{aligned} y &: \ \text{output vector, } r \times 1 \ , \\ C &: \ \text{sensor calibration matrix, } r \times n \ . \end{aligned}$$

Physically, for mechanical systems, x represents the collection of positions and velocities of the body (so for a complete description this must be twice the number of degrees of freedom), u is the various actuators (such as thrusters, rudders, propulsors), and y the outputs (what is available to us through observation or measurements).

As an example, consider the spring–mass–damper system shown in Figure 1. The equations of motion are

$$\begin{aligned} m_a \ddot{x}_a + k_a x_a + c_a \dot{x}_a + c_1(\dot{x}_a - \dot{x}_b) &= f(t) \ , \\ m_b \ddot{x}_b + k_b x_b + c_b \dot{x}_b + c_1(\dot{x}_b - \dot{x}_a) &= 0 \ . \end{aligned}$$

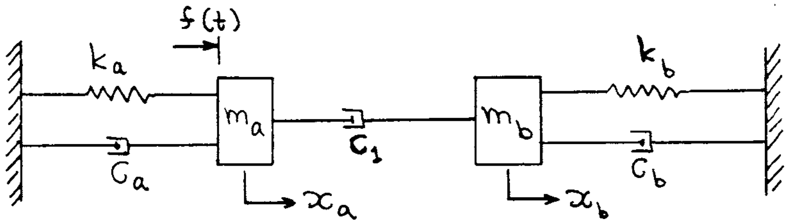

Figure 1: A spring–mass–damper system

If we take as states the position and velocity of each mass

$$\begin{aligned} x_1 &= x_a \ , \\ x_2 &= \dot{x}_a \ , \\ x_3 &= x_b \ , \\ x_4 &= \dot{x}_b \ , \end{aligned}$$

we have the equations in state form as

$$\begin{aligned} \dot{x}_1 &= x_2 \ , \\ \dot{x}_2 &= -\frac{k_a}{m_a}x_1 - \frac{c_a + c_1}{m_a}x_2 + \frac{c_1}{m_a}x_4 + \frac{1}{m_a}f \ , \\ \dot{x}_3 &= x_4 \ , \\ \dot{x}_4 &= -\frac{k_b}{m_b}x_3 - \frac{c_b + c_1}{m_b}x_4 + \frac{c_1}{m_b}x_2 \ , \end{aligned}$$

and the A, B matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_a}{m_a} & -\frac{c_a + c_1}{m_a} & 0 & \frac{c_1}{m_a} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{c_1}{m_b} & -\frac{k_b}{m_b} & -\frac{c_b + c_1}{m_b} \end{bmatrix} ,$$

and

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_a} \\ 0 \\ 0 \end{bmatrix} .$$

It should be emphasized that here we treat the external force f as our control input, this is of course legitimate if we *can* and are *willing* to change f at will so that we can affect the response of the system. This is not always the case of course; there are external forces that affect a given system and they act despite our will or even knowledge. These are called *disturbances*, and a more general form of the state equations is

$$\dot{x} = Ax + Bu + \Gamma w \ ,$$

where

$$
\begin{aligned}
w &: \text{ disturbance vector, } d \times 1 \, , \\
\Gamma &: \text{ disturbance distribution matrix, } n \times d \, .
\end{aligned}
$$

The above equations are linear; many dynamical systems, however, yield nonlinear equations of motion. The control design problem is significantly simplified when dealing with linear equations and in such a case we need to linearize the original nonlinear equations about a nominal operating point. This nominal point is physically defined usually by the designer and, roughly speaking, should be the condition where the system is expected to spend most of its life at. Usually, this is some sort of static equilibrium of the system which corresponds to a specified value for the control effort.

To formalize things say we have a nonlinear system of state equations

$$\dot{x} = f(x, u) \, .$$

Fix the control vector $u = u_0$, then

$$\dot{x} = f(x, u_0) \, .$$

Solve the nonlinear coupled algebraic set of equations

$$f(x, u_0) = 0 \, ,$$

to get the solution $x = x_0$. This is our nominal point, and solution of this set of equations is the most difficult part of the linearization process. Once x_0 has been obtained, we linearize $\dot{x} = f(x, u)$ around the nominal point $(x, u) = (x_0, u_0)$. To do this we expand in Taylor series and keep the first order terms only,

$$f(x, u) = \left. \frac{\partial f}{\partial x} \right|_{(x_0, u_0)} (x - x_0) + \left. \frac{\partial f}{\partial u} \right|_{(x_0, u_0)} (u - u_0) \, .$$

Then by assuming the change in coordinates

$$
\begin{aligned}
x &\rightarrow x - x_0 \, , \\
u &\rightarrow u - u_0 \, ,
\end{aligned}
$$

the linearized system becomes

$$\dot{x} = Ax + Bu \, ,$$

where A and B are the constant Jacobian matrices of partial derivatives evaluated at the nominal point (x_0, u_0)

$$
\begin{aligned}
A &= \left. \frac{\partial f}{\partial x} \right|_{(x_0, u_0)} \, , \\
B &= \left. \frac{\partial f}{\partial u} \right|_{(x_0, u_0)} \, .
\end{aligned}
$$

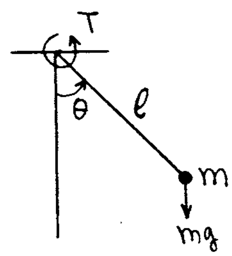

Figure 2: A simple pendulum

The elements of A are given by

$$A = [a_{ij}] \; , \quad \text{where} \quad a_{ij} = \frac{\partial f_i}{\partial x_j} \; ,$$

and similarly for B.

As an example, consider the simple pendulum shown in Figure 2. The equation of motion is

$$m\ell^2\ddot{\theta} + mg\ell \sin\theta = T \; ,$$

or

$$\ddot{\theta} + \omega_n^2 \sin\theta = \frac{T}{m\ell^2} \; , \quad \omega_n^2 = \frac{g}{\ell} \; .$$

Select as state variables

$$\begin{aligned} x_1 &= \omega_n \theta \; , \\ x_2 &= \dot{\theta} \; . \end{aligned}$$

The state equations are then

$$\begin{aligned} \dot{x}_1 &= \omega_n x_2 \; , \\ \dot{x}_2 &= -\omega_n^2 \sin\left(\frac{x_1}{\omega_n}\right) + \frac{T}{m\ell^2} \; . \end{aligned}$$

For equilibrium (with no excitation, $T = 0$)

$$\begin{aligned} \sin\frac{x_1}{\omega_n} = 0 \quad &\Rightarrow \quad (x_1)_0 = 0 \quad \text{or} \quad (x_1)_0 = \pi\omega_n \; , \\ \omega_n x_2 = 0 \quad &\Rightarrow \quad (x_2)_0 = 0 \; . \end{aligned}$$

If we choose the down position to linearize we get

$$\sin\frac{x_1}{\omega_n} = \frac{x_1}{\omega_n}$$

and the linearized equations are

$$\begin{aligned} \dot{x}_1 &= \omega_n x_2 \; , \\ \dot{x}_2 &= -\omega_n x_1 + \frac{T}{m\ell^2} \; , \end{aligned}$$

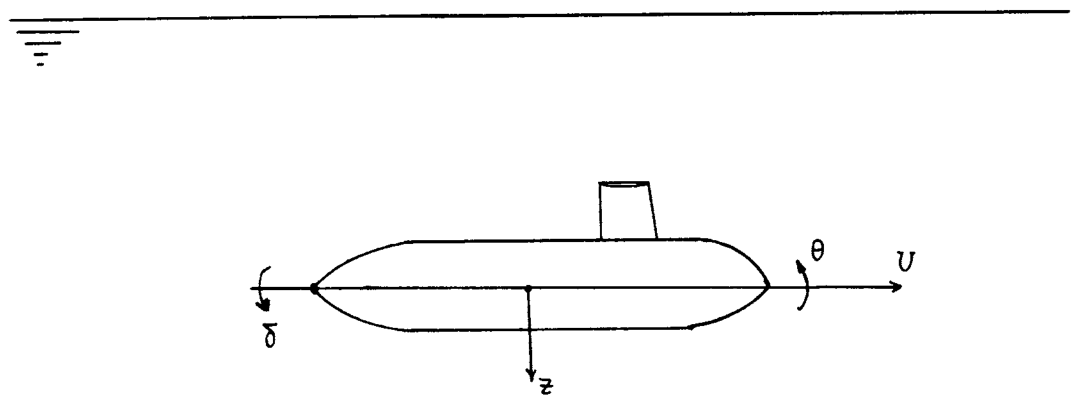

Figure 3: Variables definition for the submarine example

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{m\ell^2} \end{bmatrix}}_{B} \underbrace{T}_{u} \ .$$

Example: Consider the following equations of motion for a submarine in the dive plane (refer to Figure 3)

$$\begin{aligned}
&(m - Z_{\dot{w}})\dot{w} - (Z_{\dot{q}} + m x_G)\dot{q} = Z_w U w + (Z_q + m)Uq + m z_G q^2 \\
&\qquad +(W - B)\cos\theta + Z_\delta U^2 \delta \ , \\
&(I_y - M_{\dot{q}})\dot{q} - (M_{\dot{w}} + m x_G)\dot{w} = M_w U w + (M_q - m x_G)Uq \\
&\qquad -(x_G W - x_B B)\cos\theta - (z_G W - z_B B)\sin\theta - m z_G w q + M_\delta U^2 \delta \ , \\
&\dot{\theta} = q \ , \\
&\dot{z} = -U \sin\theta + w \cos\theta \ ,
\end{aligned}$$

where

$$\begin{aligned}
U &= \text{forward speed} \ , \\
w &= \text{heave velocity} \ , \\
q &= \text{pitch rate} \ , \\
\theta &= \text{pitch angle} \ , \\
\delta &= \text{dive plane angle} \ , \\
z &= \text{depth} \ , \\
W &= \text{weight} \ , \\
B &= \text{buoyancy} \ , \\
m &= \text{mass} \ ,
\end{aligned}$$

$$\begin{aligned}
I_y &= \text{mass moment of inertia },\\
(x_G, z_G) &= \text{coordinates of center of gravity },\\
(x_B, z_B) &= \text{coordinates of center of buoyancy },\\
Z_w &= \text{heave force hydrodynamic coefficient },\\
M_q &= \text{pitch moment hydrodynamic coefficient }.
\end{aligned}$$

Now say we want to linearize these equations for a level flight path when the dive plane angle is zero, $\delta_0 = 0$. Then by setting all time derivatives to zero (this corresponds to equilibrium) we get

$$\begin{aligned}
Z_w U w_0 + (W - B)\cos\theta_0 &= 0 ,\\
M_w U w_0 - (x_G W - x_B B)\cos\theta_0 - (z_G W - z_B B)\sin\theta_0 &= 0 ,\\
q_0 &= 0 ,\\
-U\sin\theta_0 + w_0\cos\theta_0 &= 0 .
\end{aligned}$$

If we assume that the boat is neutrally buoyant $x_G = x_B$ and $W = B$, we have

$$\begin{aligned}
Z_w U w_0 &= 0 ,\\
M_w U w_0 - (z_G - z_B) B \sin\theta_0 &= 0 ,\\
-U\sin\theta_0 + w_0\cos\theta_0 &= 0 ,
\end{aligned}$$

from which we can get the nominal position

$$w_0 = q_0 = 0 , \quad \text{and} \quad \sin\theta_0 = 0 ,$$

which means

$$\theta_0 = 0 , \quad \text{or} \quad \theta_0 = \pi .$$

These correspond to the two possible static equilibrium positions, like a regular or like an inverted pendulum.

If we choose to linearize around the $\theta_0 = 0$ equilibrium we have

$$\begin{aligned}
q^2 &= (2q_0)q = 0 ,\\
wq &= (w_0)q + (q_0)w = 0 ,\\
\sin\theta &= (\cos\theta_0)\theta = \theta ,\\
w\cos\theta &= (-w_0\sin\theta_0)\theta + (\cos\theta_0)w = w .
\end{aligned}$$

The linear equations of motion are then written as

$$\begin{aligned}
&(m - Z_{\dot w})\dot w - (Z_{\dot q} + m x_G)\dot q = Z_w U w + (Z_q + m)Uq + Z_\delta U^2 \delta ,\\
&(I_y - M_{\dot q})\dot q - (M_{\dot w} + m x_G)\dot w = M_w U w + (M_q - m x_G)Uq - (z_G - z_B)W\theta + M_\delta U^2 \delta ,\\
&\dot\theta = q ,\\
&\dot z = -U\theta + w .
\end{aligned}$$

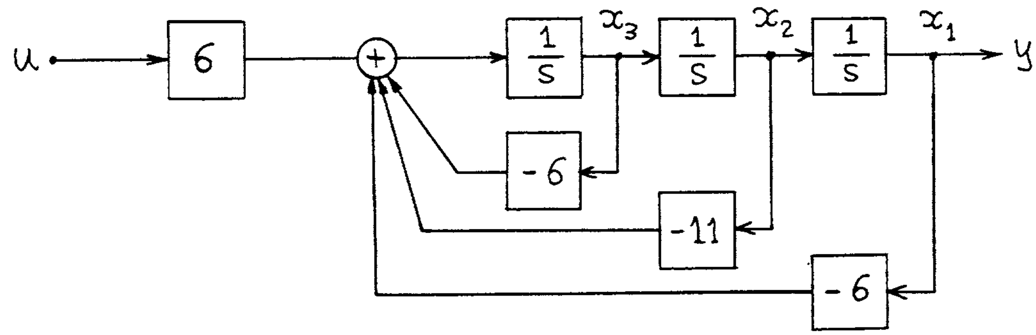

Figure 4: State equations from block diagram

In state space form these are written as

$$\underbrace{\begin{bmatrix} \dot{\theta} \\ \dot{w} \\ \dot{q} \\ \dot{z} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ a_{13} z_{GB} & a_{11} U & a_{12} U & 0 \\ a_{23} z_{GB} & a_{21} U & a_{22} U & 0 \\ -U & 1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \theta \\ w \\ q \\ z \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ b_1 U^2 \\ b_2 U^2 \\ 0 \end{bmatrix}}_{B} \underbrace{\delta}_{u} ,$$

where the coefficients a_{ij}, b_i are given by

$$\begin{aligned} D_v &= (m - Z_{\dot{w}})(I_y - M_{\dot{q}}) - (m x_G + Z_{\dot{q}})(m x_G + M_{\dot{w}}) , \\ a_{11} D_v &= (I_y - M_{\dot{q}}) Z_w + (m x_G + Z_{\dot{q}}) M_w , \\ a_{12} D_v &= (I_y - M_{\dot{q}})(m + Z_q) + (m x_G + Z_{\dot{q}})(M_q - m x_G) , \\ a_{13} D_v &= -(m x_G + Z_{\dot{q}}) W , \\ b_1 D_v &= (I_y - M_{\dot{q}}) Z_\delta + (m x_G + Z_{\dot{q}}) M_\delta , \\ a_{21} D_v &= (m - Z_{\dot{w}}) M_w + (m x_G + M_{\dot{w}}) Z_w , \\ a_{22} D_v &= (m - Z_{\dot{w}})(M_q - m x_G) + (m x_G + M_{\dot{w}})(m + Z_q) , \\ a_{23} D_v &= -(m - Z_{\dot{w}}) W , \\ b_2 D_v &= (m - Z_{\dot{w}}) M_\delta + (m x_G + M_{\dot{w}}) Z_\delta , \end{aligned}$$

and $z_{GB} = z_G - z_B$ is the metacentric height. We will use the above equations of motion as our main example case in these notes. It should be noted that the equations correspond to Swimmer Delivery Vehicle 17.5 feet in length. This is not needed in the calculations that follow but it gives an idea of the sizes involved. One thing we have to emphasize is that in the submarine examples in these notes U is the forward speed (not control). The control is designated by δ; this is standard notation (see ME 4823 for more details).

1.2 From Block Diagrams to State Equations

The transition between block diagram form (what we were using in ME 3801) and state equations (what we are using in ME 4811) is relatively simple and can be divided into a series of different cases.

1. State equations from block diagram

Suppose we have the block diagram shown in Figure 4, and we want to write a set of state equations for this system. We observe that the system is third order (it has three integrators, so its characteristic equation will be third order). Therefore, we need three state equations and three states. One choice is to take as states the outputs of the integrator blocks. This way we get

$$\begin{aligned}\dot{x}_1 &= x_2 \ , \\ \dot{x}_2 &= x_3 \ , \\ \dot{x}_3 &= -6x_1 - 11x_2 - 6x_3 + 6u \ ,\end{aligned}$$

and the output equation

$$y = x_1 \ .$$

The A, B, and C matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} .$$

We note that the above choice of states is not unique, we could have selected as states the outputs of the three feedback blocks; this would have produced a different but equivalent (with the same input–output relationship) system of state equations.

2. Block diagram from state equations

Consider the following system of state equations

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_1 u \ , \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_2 u \ , \\ y &= c_1 x_1 + c_2 x_2 \ .\end{aligned}$$

The A, B, C matrices here are

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} , \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} , \quad C = \begin{bmatrix} c_1 & c_2 \end{bmatrix} .$$

The block diagram is constructed as shown in Figure 5.

3. Block diagram and state equations from differential equation

Consider the transfer function between input u and output y

$$\frac{y}{u} = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} ,$$

which is equivalent to the differential equation

$$y^{(iii)} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{u} + b_0 u \ .$$

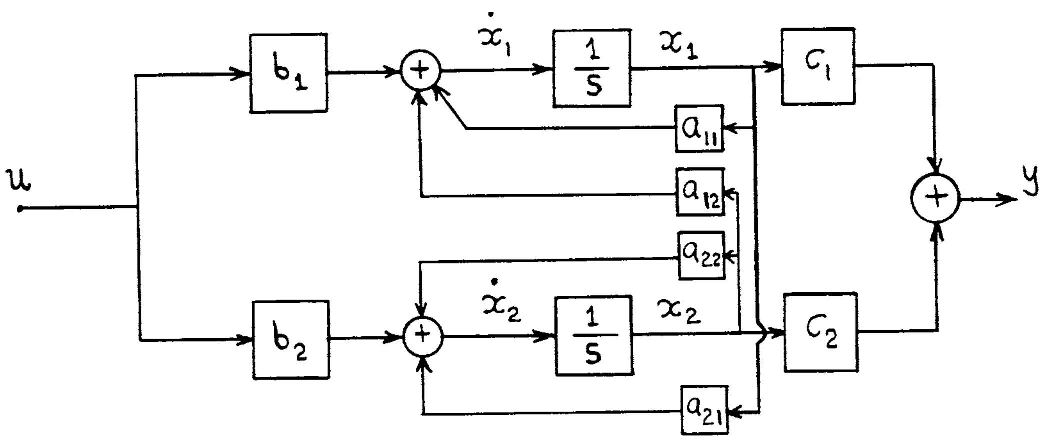

Figure 5: Block diagram from state equations

This is a third order system, so we need three states. Let our first state be

$$x_1 = y \ ,$$

so

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

Substitute $x_1 = y$ into the equation,

$$x_1^{(iii)} + a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0 y = b_1\dot{u} + b_0 u \ .$$

To lower the order let

$$\dot{x}_1 = x_2 \ , \quad \text{this is our first state equation}$$

and substitute again

$$\ddot{x}_2 + a_2\dot{x}_2 + a_1 x_2 + a_0 x_1 = b_1\dot{u} + b_0 u \ .$$

Now if we substitute $x_3 = \dot{x}_2$ we see that the $\dot{u}$ term in the equation will survive, and this goes against our general state space form $\dot{x} = Ax + Bu$. To eliminate the $\dot{u}$ term we substitute

$$\begin{aligned} x_3 &= \dot{x}_2 - b_1 u \quad \text{or} \\ \dot{x}_2 &= x_3 + b_1 u \quad \text{this is our second state equation} \end{aligned}$$

One more substitution will then result in

$$\dot{x}_3 + b_1\dot{u} + a_2 x_3 + a_2 b_1 u + a_1 x_2 + a_0 x_1 = b_1\dot{u} + b_0 u \ ,$$

or

$$\dot{x}_3 = -a_2 x_3 - a_1 x_2 - a_0 x_1 + (b_0 - a_2 b_1)u \ ,$$

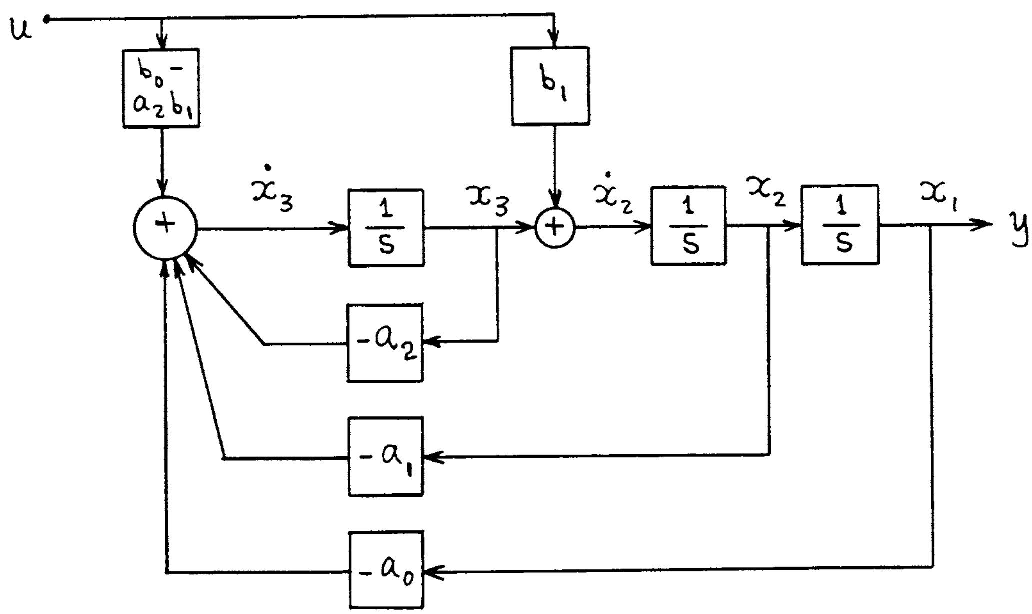

Figure 6: Block diagram and state equations from differential equation

which is the third state equation.

The state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \\ b_0 - a_2 b_1 \end{bmatrix} u ,$$

and the output equation

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

The above form of the A matrix is called a companion form (negative coefficients in the last row, and ones in the superdiagonal).

The block diagram appears as shown in Figure 6.

1.3 From State Equations to Transfer Function

Consider the standard state space system

$$\begin{aligned} \dot{x} &= Ax + Bu , \\ y &= Cx . \end{aligned}$$

In the Laplace domain (with zero initial conditions) this becomes

$$sX(s) = AX(s) + BU(s) ,$$

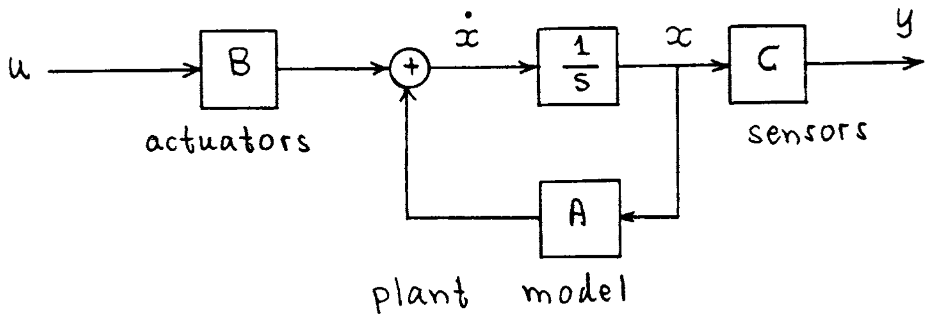

Figure 7: A generic block diagram

$$Y(s) = CX(s) ,$$

or

$$(sI - A)X = BU \Longrightarrow X = (sI - A)^{-1}BU ,$$
$$Y = C(sI - A)^{-1}BU .$$

If we compare the last expression with

$$Y(s) = G(s)U(s) , \quad \text{where } G(s) \text{ is the transfer function}$$

we can see that

$$G(s) = C(sI - A)^{-1}B ,$$

is the transfer function of the system. This is of the familiar ME 3801 form only in the case of a single input single output (SISO) system (i.e., both u and y are scalars instead of vectors). In the more general case of a multiple input multiple output system (MIMO), it is a transfer function matrix and its individual elements consist of transfer functions in the usual sense. It can be thought of as a matrix of influence coefficients (the ij element of the matrix depicts the transfer function between the i–th output and the j–input).

The above helps in constructing compact generic block diagrams, as shown in Figure 7.

$$\dot{x} = Ax + Bu , \quad y = Cx$$

1.4 Poles and Zeros

Recall that for a system in the form

$$\dot{x} = Ax + Bu , \qquad y = Cx$$

its transfer function is written as

$$G(s) = C(sI - A)^{-1}B .$$

The *poles* of the transfer function are defined as those values of s where the denominator goes to zero. This means that

$$\begin{aligned}
&(sI - A) \text{ is a singular matrix, or} \\
&\det[sI - A] = 0 \text{ or} \\
&s = \text{ eigenvalue of } A \; .
\end{aligned}$$

The *zeros* of the transfer function are usually defined for SISO systems. In such a case we have

$$G(s) = \det\left[C(sI - A)^{-1}B\right] \; ,$$

and using properties of the determinant we get

$$\begin{aligned}
\det[C(sI - A)^{-1}B] &= \frac{\det[sI - A] \cdot \det[C(sI - A)^{-1}B]}{\det[sI - A]} \\
&= \frac{\det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}}{\det[sI - A]}
\end{aligned}$$

where we used the fact that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \cdot \det[D - CA^{-1}B] \; .$$

Therefore, the zeros of $G(s)$ are solutions of

$$\det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = 0 \; .$$

As an example, say we have the system

$$\begin{aligned}
\dot{x}_1 &= -3x_1 + x_2 + u \; , \\
\dot{x}_2 &= 2u \; , \\
y &= x_1 \; .
\end{aligned}$$

The matrices A, B, C are

$$A = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \; , \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \; , \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \; .$$

The poles of the system are

$$\det[sI - A] = \det\begin{bmatrix} s+3 & 1 \\ 0 & s \end{bmatrix} = s(s+3) = 0 \Longrightarrow s = 0, -3 \; ,$$

and the zeros

$$\det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = \det\begin{bmatrix} s+3 & 1 & -1 \\ 0 & s & -2 \\ 1 & 0 & 0 \end{bmatrix} = 2 + s = 0 \Longrightarrow s = -2 \; .$$

To verify this, let's get $G(s)$ using classical methods:

$$\begin{aligned}
&\dot{y} = -3y + x_2 + u \ , \quad \text{or} \\
&\ddot{y} = -3\dot{y} + 2u + \dot{u} \ , \quad \text{or} \\
&\ddot{y} + 3\dot{y} = \dot{u} + 2u \ , \quad \text{or} \\
&Y(s^2 + 3s) = U(s + 2) \ , \quad \text{or} \\
&\frac{Y(s)}{U(s)} = \frac{s+2}{s(s+3)} \ ,
\end{aligned}$$

which agrees with the poles and zeros from state space. These poles and zeros are usually called *open loop* poles and zeros since no feedback control action has been defined yet.

Example: Consider the state equations for the submarine example, where the state vector is

$$x = [\theta, w, q] \ ,$$

the output vector is the pitch angle

$$y = \theta \ ,$$

and the control input u is the dive plane angle δ

$$u = \delta \ .$$

The state equations are the same as before. Typical values for the coefficients are

$$\begin{aligned}
&a_{11} = -0.064390823 \ , \quad a_{12} = -0.1420481 \ , \quad a_{13} = 0.1353290 \ , \\
&a_{21} = 0.025208820 \ , \quad a_{22} = -0.1479027 \ , \quad a_{23} = -0.3599404 \ , \\
&b_1 = 0.0012883232 \ , \quad b_2 = -0.0034266096 \ , \\
&z_{GB} = 0.1\text{ft} \ , \quad U = 5\text{ft/sec} \ .
\end{aligned}$$

Using MATLAB and the above values we can find the transfer function

$$\frac{\theta}{\delta} = \frac{-0.0857s - 0.0235}{s^3 + 1.0615s^2 + 0.3636s + 0.0099} \ ,$$

and we can see that the open loop poles are simply the roots of the denominator polynomial

$$-0.5159 \pm 0.2584i \ , \quad -0.0297 \ .$$

These are also given by the eigenvalues of matrix A. Notice that the system is open loop stable. This means that with no control action δ, if an initial disturbance is introduced in the angle θ, it will go back to zero asymptotically. As the metacentric height z_{GB} gets closer to zero, one open loop pole goes to zero. (Can you see this from the form of the A matrix? What is the physical significance of a zero pole?) The open loop zero is the root of the numerator of the transfer function

$$-0.2742 \ .$$

The transfer function can also be computed by starting with the equations of motion

$$\begin{aligned}\dot{\theta} &= q \ , \\ \dot{w} &= a_{13} z_{GB} \theta + a_{11} U w + a_{12} U q + b_1 U^2 \delta \ , \\ \dot{q} &= a_{23} z_{GB} \theta + a_{21} U w + a_{22} U q + b_2 U^2 \delta \ ,\end{aligned}$$

constructing the block diagram from δ to θ, and reducing it, as we did in Section 1.2.

1.5 Time Response Using State Equations

There are two ways to compute the time response of a system using the state equations: numerical and analytical.

1. Numerical

State equations are naturally used in digital computer simulation. For example, if we use Euler's integration: given $x(0)$ and $u(0)$ at $t = 0$, then

$$x(t + \Delta t) = x(t) + \dot{x}(t)\, \Delta t \ .$$

Δt is the integration time step which must be selected small enough (with respect to the natural time constant of the system) for results to be valid; and $\dot{x}(t) = Ax(t) + Bu(t)$, in other words we evaluate $\dot{x}$ using the current value of x and u. Continuing the scheme, we get

$$\begin{aligned}x(\Delta t) &= x(0) + [Ax(0) + Bu(0)]\, \Delta t \ , \\ x(2\Delta t) &= x(\Delta t) + [Ax(\Delta t) + Bu(\Delta t)]\, \Delta t \ ,\end{aligned}$$

and so on. Although Euler's method is the simplest and most inaccurate numerical integration technique available, it is good enough for naval engineering problems where things do not change very fast in time.

2. Analytical

We want the transient solution for

$$\dot{x} = Ax \ , \quad x(t_0) = x(0) \ ,$$

where x is the $n \times 1$ state vector, A is the $n \times n$ open loop dynamics matrix, and $x(0)$ is the $n \times 1$ vector of initial conditions. Recall that for a first–order system ($n = 1$) we would have

$$\dot{x} = ax \ , \quad x(t_0) = x(0) \ .$$

If we assume

$$x = \alpha e^{st} \ ,$$

we get

$$\begin{aligned}&\dot{x} - ax = 0 \ \text{ or} \\ &\alpha e^{st}(s - a) = 0 \ \text{ or} \\ &s = a \ , \quad \text{an eigenvalue} \ .\end{aligned}$$

Therefore, the solution is

$$x = \alpha e^{at} .$$

The unknown constant α can be computed from the initial condition

$$x(t_0) = \alpha e^{at_0} = x(0) ,$$

giving

$$\alpha = x(0)e^{-at_0} .$$

The solution is then

$$x(t) = e^{a(t-t_0)}x(0) ,$$

where

$$e^{a(t-t_0)} = 1 + \frac{a(t-t_0)}{1!} + \frac{[a(t-t_0)]^2}{2!} + \frac{[a(t-t_0)]^3}{3!} + \cdots$$

When the solution is extended to a matrix system ($n > 1$), the results are completely parallel,

$$\dot{x} = Ax ,$$

with solution

$$\underbrace{x(t)}_{\text{vector}} = \underbrace{e^{A(t-t_0)}}_{\text{matrix}}\underbrace{x(0)}_{\text{vector}} ,$$

where the matrix exponential is defined through a series expansion analogously to its scalar counterpart

$$e^{A(t-t_0)} = I + \frac{A(t-t_0)}{1!} + \frac{[A(t-t_0)]^2}{2!} + \frac{[A(t-t_0)]^3}{3!} + \cdots$$

This is called the *state transition matrix* denoted by

$$\Phi(t-t_0) \equiv e^{A(t-t_0)} .$$

The state transition matrix expresses how the state is changed from its value at t_0 to the state at t by the system with open loop dynamics given by A

$$x(t) = \Phi(t-t_0)x(t_0) .$$

We can obtain the complete solution with a control input $u(t)$ as:

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}\left[\underbrace{\dot{x}(t)}_{\dot{x}(t)=Ax+Bu} - Ax(t)\right] = e^{-At}Bu(t) .$$

Integrating,

$$e^{-At}x(t) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau + c ,$$

where c is a vector constant of integration. Now at $t = t_0$ we have

$$e^{-At_0}x(0) = c ,$$

giving

$$e^{-At}x(t) = \int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau + e^{-At_0}x(0)\ .$$

Multiplying through by e^{At}

$$x(t) = e^{A(t-t_0)}x(0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau\ , \quad t \geq t_0\ ,$$

or

$$x(t) = \underbrace{\Phi(t-t_0)x(0)}_{\text{transient}} + \underbrace{\int_{t_0}^{t} \Phi(t-\tau)Bu(\tau)\, d\tau}_{\text{steady state}}\ .$$

In most cases

$$\text{transient} = \text{response due to initial state}$$

and this will go to zero for a stable system, while

$$\text{steady state} = \text{response due to input}$$

is given by the above convolution integral. For linear systems, the total response is of course the sum of the two responses.

The matrix exponential e^{At} can be computed using a couple of different ways.

- One way is with the above power series expansion

$$e^{At} = I + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots\ .$$

  This is efficient only numerically when the series can be truncated to an arbitrary degree of accuracy. In general, these Taylor series are used to define rather than to compute functions of a matrix (take a 2×2 matrix and try to find its cosine using the appropriate series expansion; then check your answer using MATLAB).

- If A can be diagonalized; i.e., if $\Lambda = T^{-1}AT$ where T is the matrix of eigenvectors of A and Λ the diagonal matrix of the eigenvalues of A,

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\ ,$$

  then

$$e^{At} = T^{-1}e^{\Lambda t}T\ ,$$

  where

$$e^{\Lambda t} = \text{diag}\left\{e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}\right\}\ .$$

  We can easily see from the last expression why if at least one of the eigenvalues λ_i of A is positive, the system will be unstable.

For time varying systems of the form

$$\dot{x} = A(t)x \ ,$$

the state transition matrix is denoted by

$$\Phi(t, t_0) \ ,$$

and the solution is given by

$$x(t) = \Phi(t, t_0)x(0) \ .$$

Notice that the state transition matrix for time varying systems is function of both the current time t and initial time t_0, unlike the time invariant system case where Φ was a function of one variable only, $t - t_0$, the time interval between t and t_0. What is more unfortunate is the fact that closed form expression for $\Phi(t, t_0)$ does not exist which makes analysis and control of time varying systems much more difficult than time invariant systems considered here. As a word of caution, in general,

$$\Phi(t, t_0) \neq e^{\int_{t_0}^{t} A(\tau)d\tau} \ ,$$

except when the matrices $A(t)$ and $\int A(t)dt$ commute; i.e., when

$$A(t)\left(\int A(t)dt\right) = \left(\int A(t)dt\right)A(t) \ .$$

Some general properties of the state transition matrix $\Phi(t, t_0)$ are

1. It satisfies the differential equation with identity initial conditions,

$$\begin{aligned} \dot{\Phi}(t, t_0) &= A(t)\Phi(t, t_0) \ , \\ \Phi(t_0, t_0) &= I \ . \end{aligned}$$

2. It satisfies the semi–group property,

$$\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0) \ .$$

3. It is always nonsingular,

$$\Phi^{-1}(t, t_0) = \Phi(t_0, t) \ .$$

4. It has a computable determinant,

$$\det \Phi(t, t_0) = e^{\int_{t_0}^{t} \mathrm{trace} A(\tau)d\tau} \ .$$

The main advantages of using the state transition matrix in system dynamics are two:

- Helps in proving other theorems.

- Once it has been determined, it makes calculation of the particular solution in response to some initial conditions and input, much faster.

In general, the analytic method of solution is employed only for theoretic purposes or in special circumstances; in almost all cases we obtain the solutions numerically. This has the added advantage that it is not restricted to linear systems, nonlinear systems can be simulated numerically in much the same way.

Example: Consider the submarine linear equations of motion

$$\begin{aligned}\dot{\theta} &= q \ , \\ \dot{w} &= a_{13} z_{GB} \theta + a_{11} U w + a_{12} U q + b_1 U^2 \delta \ , \\ \dot{q} &= a_{23} z_{GB} \theta + a_{21} U w + a_{22} U q + b_2 U^2 \delta \ ,\end{aligned}$$

where we assume a dive plane deflection $\delta = -0.2$ radians (-11.5 degrees). A simulation algorithm using Euler's integration is as follows:

- Step 1: Choose integration time step Δt and initial conditions θ_0, w_0, q_0. Set $i = 0$.
- Step 2: Using the values of θ_i, w_i, q_i, compute $\dot{\theta}_i$, $\dot{w}_i$, $\dot{q}_i$ from the equations of motion.
- Step 3: Compute

$$\begin{aligned}\theta_{i+1} &= \theta_i + \dot{\theta}_i \cdot \Delta t \ , \\ w_{i+1} &= w_i + \dot{w}_i \cdot \Delta t \ , \\ q_{i+1} &= q_i + \dot{q}_i \cdot \Delta t \ .\end{aligned}$$

- Step 4: Set $i = i + 1$ and go back to Step 2.

Typical results of the simulation in terms of the pitch angle θ are shown in Figure 8. As with any numerical results, however, the real question is: are they correct? The answer to this borders between art and science, and in the context of system simulations here is a set of a few checks:

1. In this particular simulation we used a time step $\Delta t = 0.01$ seconds. Is this small enough? The easiest way to check this is to reduce (or increase) Δt, say by a factor of 10, and re–run the program. If the results do not change, the above choice for Δt was good. A more rational way to do the same thing would be to look at the natural time constant of the dynamics of the system. The system poles were found in page 18. It seems that the fastest pole of the system has real part -0.5159, and the time constant that corresponds to this is about $1/0.5$ or 2 seconds. This means that it takes a couple of seconds for the boat to "listen" to its dive planes, so $\Delta t = 0.01$ should give very accurate results. In fact in this case we could go as far as $\Delta t = 0.5$ and we would still be reasonably accurate.

2. Look again at the system eigenvalues: one of them is certainly dominant, -0.0297, so the response should approximate that of a first order system with a time constant $1/0.0297$, or about 33.5 seconds. Now look at the response of the figure: does it take approximately 33.5 seconds to go up to 60% of its final value?

3. By now we are convinced that the transient response we see in the figure agrees with our engineering intuition. How about the final or steady state value of the response? This is something we can compute exactly. At steady state we should have, $\dot{\theta} = \dot{w} = \dot{q} = 0$, so that our equations become at steady state:

$$
\begin{aligned}
&q = 0 \ , \\
&a_{13} z_{GB} \theta + a_{11} U w + a_{12} U q + b_1 U^2 \delta \ , \\
&a_{23} z_{GB} \theta + a_{21} U w + a_{22} U q + b_2 U^2 \delta \ .
\end{aligned}
$$

   Using $q = 0$, the second and third equations give

$$
\begin{aligned}
a_{13} z_{GB} \theta + a_{11} U w &= -b_1 U^2 \delta \ , \\
a_{23} z_{GB} \theta + a_{21} U w &= -b_2 U^2 \delta \ .
\end{aligned}
$$

   Substituting $\delta = -0.2$ and using the values from page 17 we find

$$
\theta = 0.476 \text{ radians or } 27.3 \text{ degrees} \ ,
$$

   a result which agrees with the figure.

Simulation of a nonlinear set of equations proceeds in a similar manner. Let's assume that the only important nonlinearities in our example come from the trigonometric functions and not the hydrodynamic forces and moments; in other words the nonlinear equations of motion are

$$
\begin{aligned}
\dot{\theta} &= q \ , \\
\dot{w} &= a_{13} z_{GB} \sin\theta + a_{11} U w + a_{12} U q + b_1 U^2 \delta \ , \\
\dot{q} &= a_{23} z_{GB} \sin\theta + a_{21} U w + a_{22} U q + b_2 U^2 \delta \ .
\end{aligned}
$$

The numerical integration proceeds in exactly the same way as before; the only difference is that here the values for $\dot{w}$ and $\dot{q}$ are computed from the new equations. Typical results are shown in the previous figure where the difference between linear and nonlinear simulations is also shown. Naturally, whenever possible, simulations must be performed for the nonlinear systems since these model the underlying physics more accurately. The steady state value for θ can be computed from the nonlinear equations in the same way as before, the algebra is easy in the example case but keep in mind that for general nonlinear equations it may be *very* difficult. Here we can find

$$
\sin\theta = 0.476 \quad \text{or} \quad \theta = 28.5 \text{ degrees} \ .
$$

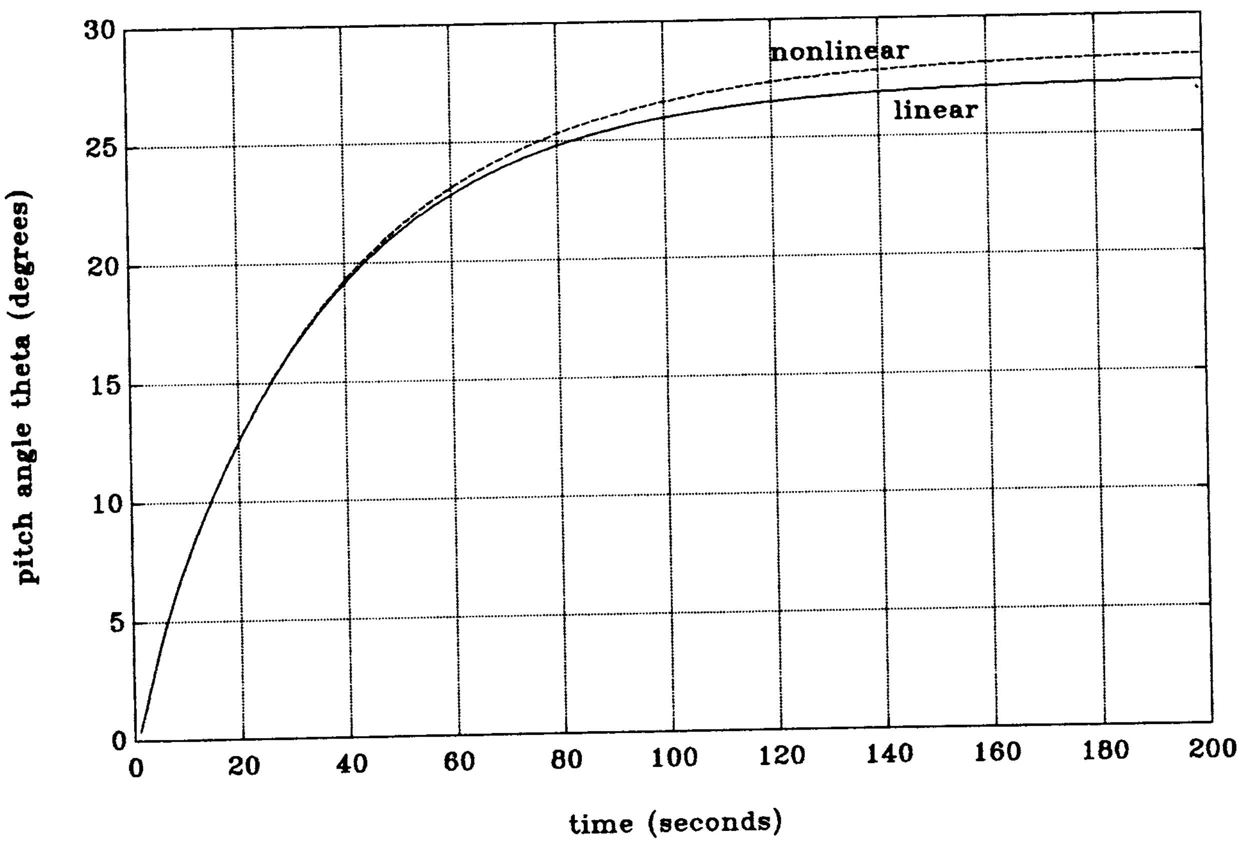

Figure 8: Response for the submarine example

1.6 Canonical Forms

Consider the general state equations

$$\begin{aligned}\dot{x} &= Ax + Bu \ , \\ y &= Cx \ .\end{aligned}$$

We can introduce a similarity transformation which will transorm the system into a new set of state variables; the eigenvalues will be unchanged:

$$\begin{aligned}x &= Tx' \ , \\ \dot{x} &= T\dot{x}' \ ,\end{aligned}$$

where x' is the new set of state variables, and T is the transformation matrix. We can substitute now into the state equations to get

$$T\dot{x}' = ATx' + Bu \ ,$$

or

$$\dot{x}' = T^{-1}ATx' + T^{-1}Bu \ ,$$

and

$$y = CTx' \ .$$

The task is to choose T such that $T^{-1}AT$ looks "nice".

If the matrix A has distinct eigenvalues λ_i with associated eigenvectors v_i, we have

$$Av_i = v_i\lambda_i \ ,$$

and we can group these together column by column to get

$$A\begin{bmatrix} v_1 & v_2 & \cdot & \cdot & \cdot & v_n \end{bmatrix} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdot & \cdot & \cdot & v_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{bmatrix}}_{\Lambda} \ .$$

T is the modal matrix of eigenvectors, and Λ is the diagonal matrix of eigenvalues of A. We then have

$$AT = T\Lambda \ , \quad \text{or} \quad T^{-1}AT = \Lambda \ .$$

If we use the modal matrix as the trensformation matrix T, we will produce the *normal coordinate form*:

$$\begin{aligned}&\dot{x}' = T^{-1}ATx' + T^{-1}Bu = \Lambda x' + B'u \ , \\ &y = CTx' = C'x' \ ,\end{aligned}$$

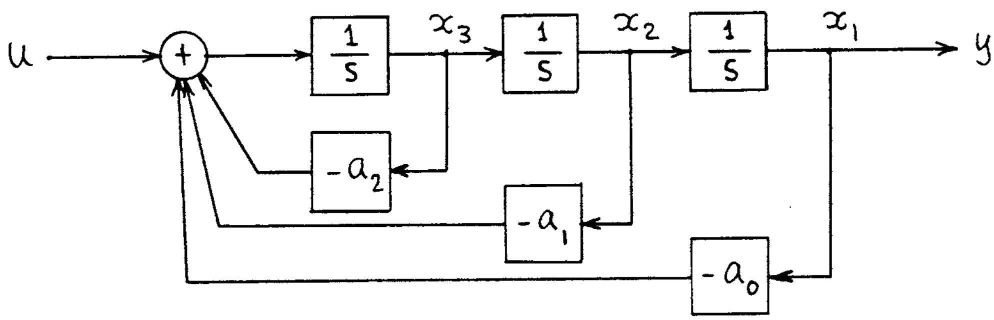

Figure 9: Block diagram in control canonical form

where

$$B' = T^{-1}B \quad \text{and} \quad C' = CT \ .$$

There are other "nice" forms possible. Two of them are particularly attractive in control systems.

Say we have a transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \ .$$

A nice state space form for this system is (verify this)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \ ,$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ .$$

The block diagram form is shown in Figure 9. This form of the A matrix is called *control canonical form*, or first companion form, and is naturally used in controller design as we will see later.

Consider the same transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \ .$$

Another nice form for this system is (verify this)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \ ,$$

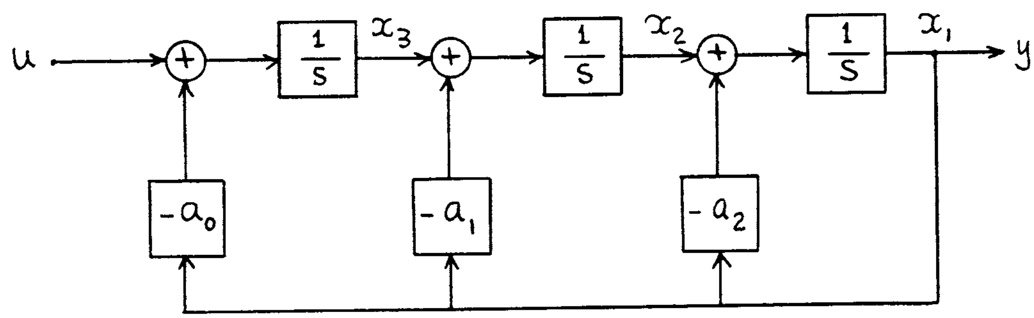

Figure 10: Block diagram in observer canonical form

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

The block diagram form is shown in Figure 10. This form of the A matrix is called *observer canonical form*, or second companion form, and is naturally used in observer design as we will see later.

More generally, assume that our transfer function is of the form

$$\frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} .$$

The control canonical form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u ,$$

and

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ,$$

with the block diagram shown in Figure 11.

The observer canonical form for the same system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u ,$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ,$$

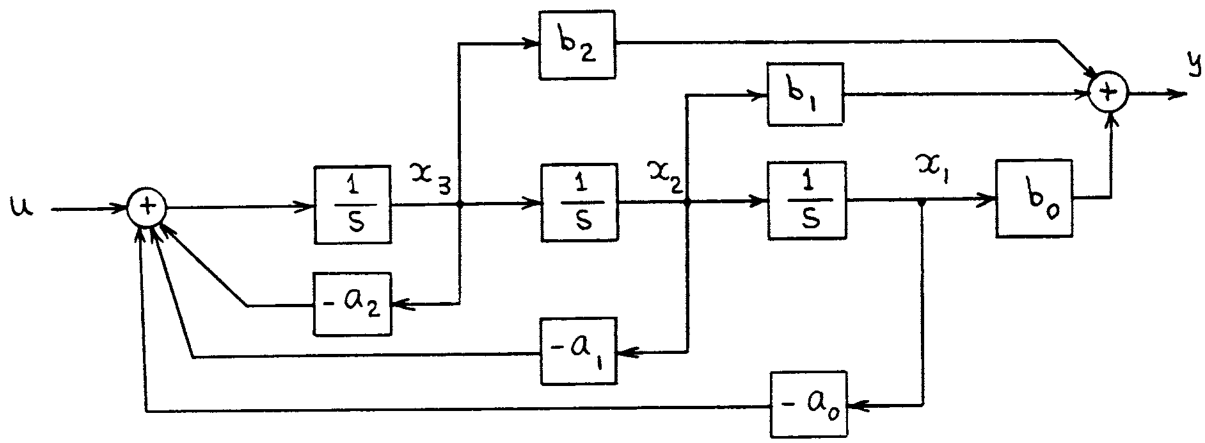

Figure 11: Block diagram in control canonical form including numerator dynamics

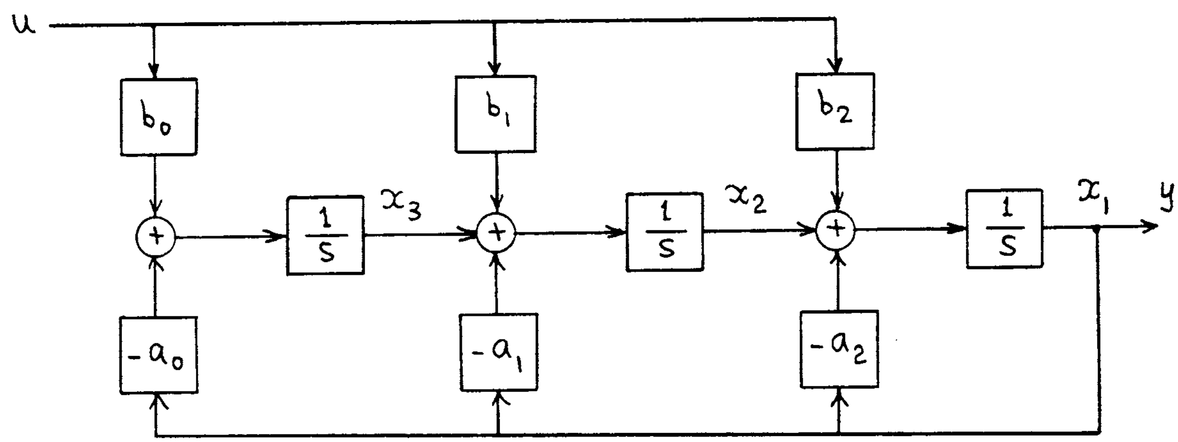

Figure 12: Block diagram in observer canonical form including numerator dynamics

with the block diagram shown in Figure 12.

You should, of course, verify the above forms! The main difference between the two forms is that in the control canonical form the B matrix is "clean", whereas in the observer canonical form it is the C matrix that appears to be "clean" instead. In both cases, observe that the characteristic equation of the A matrix can be obtained easily without any algebra. This is a very nice property of matrices in companion form and is true regardless of the order of the matrix. Finally, it should be emphasized that both forms represent exactly the same physical system; the definitions for the state are different in the two forms. In practice, one definition may make more sense than the other physically, and this is the one that should be chosen. Although defining convenient states may make the algebra simpler, it is much more preferable to choose as states variables that make sense physically; using MATLAB makes all linear algebra calculations relatively straight forward.

1.7 Controllability and Observability

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} u \ ,$$

and

$$y = \begin{bmatrix} 7 & 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} .$$

So far, the system looks nice. Let's find the transfer function:

$$\begin{aligned} G(s) &= \frac{Y(s)}{U(s)} \\ &= C(sI - A)^{-1}B \\ &= \frac{(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)} \\ &= \frac{1}{s+1} \ , \end{aligned}$$

which is first order instead of fourth as the original system, due to the multiple zero–pole cancellation. To see what went wrong, let's transform the system to its normal coordinate form by diagonalizing A. The matrix of eigenvectors of A is

$$T = \begin{bmatrix} 0.7071 & 0.4082 & 0.0000 & 0.0000 \\ -0.7071 & -0.8165 & 0.4082 & 0.0000 \\ 0.0000 & 0.4082 & -0.8165 & -0.4472 \\ 0.0000 & 0.0000 & 0.4082 & 0.8944 \end{bmatrix} .$$

Then using our familiar transformation

$$x = Tx' \quad \text{or} \quad x' = T^{-1}x \ ,$$

the system is transformed into

$$\begin{aligned} \dot{x}' &= A'x' + B'u \ , \\ y &= C'x' \ , \end{aligned}$$

where

$$\begin{aligned} A' &= T^{-1}AT = \Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \ , \\ B' &= T^{-1}B = \begin{bmatrix} 1.4142 \\ 0 \\ -2.4495 \\ 0 \end{bmatrix} \ , \\ C' &= CT = \begin{bmatrix} 0.7071 & -0.4082 & 0 & 0 \end{bmatrix} \ . \end{aligned}$$

The state equations are then

$$\begin{aligned} \dot{x}'_1 &= -x'_1 + 1.4142u \ , \\ \dot{x}'_2 &= -2x'_2 \ , \\ \dot{x}'_3 &= -3x'_3 - 2.4495u \ , \\ \dot{x}'_4 &= -4x'_4 \ , \end{aligned}$$

and the output equation

$$y = 0.7071x'_1 - 0.4082x'_2 \ .$$

In block diagram the system in normal coordinates appears as shown in Figure 13. Looking at this block diagram we can see the following

1. x'_1 : affected by the input; visible in the output;
2. x'_2 : unaffected by the input; visible in the output;
3. x'_3 : affected by the input; invisible in the output;
4. x'_4 : unaffected by the input; invisible in the output.

Therefore, it is fair to say that as far as the state variables go:

1. x'_1 : we can control it and we can observe it;
2. x'_2 : we can not control it but we can observe it;

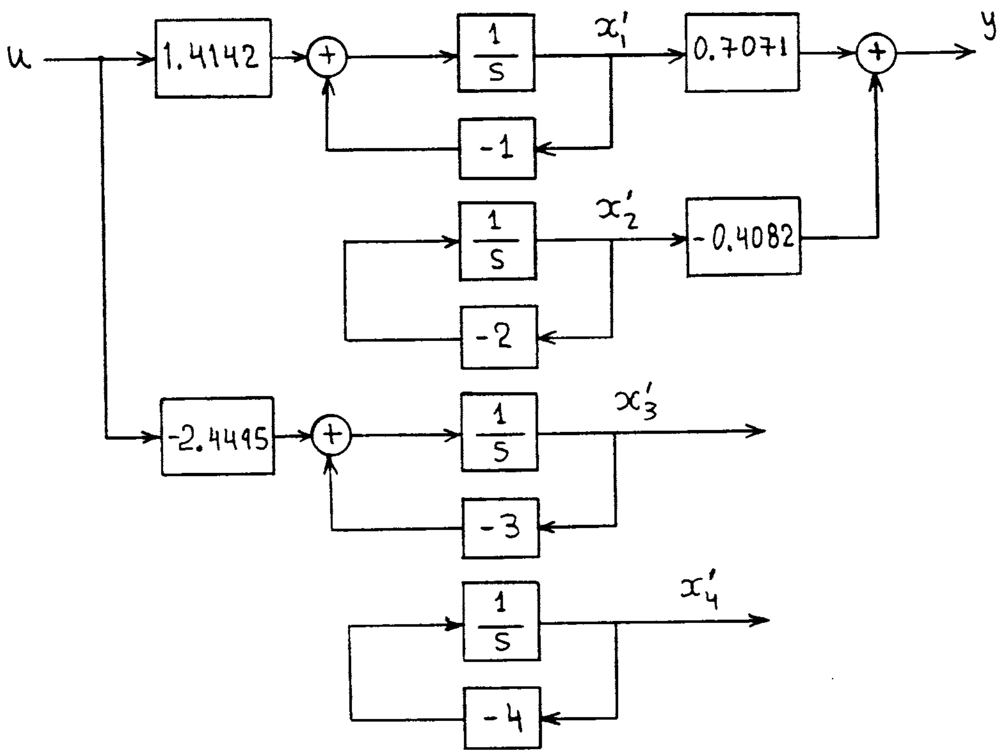

Figure 13: Block diagram illustrating uncontrollable/unobservable subsystems

3. x'_3 : we can control it but we can not observe it;

4. x'_4 : we can not control it and we can not observe it.

The final transfer function, $G(s)$, shows the first subsystem, x'_1, only.

In general, every system

$$\begin{aligned} \dot{x} &= Ax + Bu \;, \\ y &= Cx \;, \end{aligned}$$

can be divided through a series of transformations into four subsystems:

1. A controllable and observable part.

2. An uncontrollable and observable part.

3. A controllable and unobservable part.

4. An uncontrollable and unobservable part.

This is known as Kalman's decomposition theorem. The thing to remember is that the transfer function of any system is determined only by the controllable and observable subsystem. That is, the transfer function may contain less information than what is actually needed to model the complete system.

The precise definition of controllability is:

- A system is said to be state *controllable* if any initial state $x(t_0)$ can be driven to any final state $x(t_f)$ using possibly unbounded control $u(t)$ in finite time $t_0 < t < t_f$.

From the state equations

$$\dot{x} = \underbrace{A}_{n \times n} x + Bu \;,$$

this should depend only on A and B. The test for controllability is as follows: Compute the

$$\text{controllability matrix} \quad \mathcal{C} = \left[B, AB, A^2B, \ldots, A^{n-1}B\right] \;,$$

and the system is controllable if and only if the rank of $\mathcal{C}$ (the number of linearly independent rows or columns) is n. Roughly speaking, $\mathcal{C}$ shows how possible it is to change the state of a system using the input. For a single input system B is $n \times 1$ and $\mathcal{C}$ is a square matrix. The test is then that $\mathcal{C}$ be nonsingular

$$\det \mathcal{C} \neq 0 \;.$$

We can also test controllability by transforming to the normal coordinate form (with distinct eigenvalues). The system is then controllable if $B' = T^{-1}B$ has no zero row.

Example: Consider the submarine equations of motion

$$\begin{bmatrix} \dot{\theta} \\ \dot{w} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a_{13} z_{GB} & a_{11} U & a_{12} U \\ a_{23} z_{GB} & a_{21} U & a_{22} U \end{bmatrix} \begin{bmatrix} \theta \\ w \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 U^2 \\ b_2 U^2 \end{bmatrix} \delta \ ,$$

and substituting the values for the coefficients

$$\begin{bmatrix} \dot{\theta} \\ \dot{w} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0.0135 & -0.3220 & -0.7102 \\ -0.0360 & 0.1260 & -0.7395 \end{bmatrix} \begin{bmatrix} \theta \\ w \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0322 \\ -0.0857 \end{bmatrix} \delta \ .$$

The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & -0.0857 & 0.0674 \\ 0.0322 & 0.0505 & -0.0653 \\ -0.0857 & 0.0674 & -0.0404 \end{bmatrix} \ ,$$

which is full rank, 3. Therefore, the system is controllable and we can change any state θ, w, or q using the dive planes at will. Note, however, that some changes may be impractical or even impossible in practice; for example, even if the system is controllable it is not feasible to change the pitch angle to, say, 90 degrees! This would require an enormous dive plane strength which is not available in practice.

The definition for observability is

- A system is *observable* if any value of the state $x(t_0)$ can be exactly determined using a set of measurements over a finite period $t_0 < t < t_f$.

Observability depends on A and C only, and the test is: Compute the

$$\text{observability matrix} \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} \ ,$$

and the system is observable if and only if the rank of $\mathcal{O}$ is n. Roughly speaking, $\mathcal{O}$ shows how possible it is to reconstruct the state, x, of a system using a limited set of measurements, y. For a single output case C is $1 \times n$ and $\mathcal{O}$ is a square matrix. The test is then that $\mathcal{O}$ be nonsingular

$$\det \mathcal{O} \neq 0 \ .$$

We can also test observability by transforming the system to the normal coordinate form (with distinct eigenvalues). The system will then be observable if $C' = CT$ has no zero column.

Example: Consider the previous submarine equations of motion, and assume that the only sensor aboard measures the pitch angle, θ. The measurement equation is

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ w \\ q \end{bmatrix} .$$

Using A and C, the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -0.0360 & 0.1260 & -0.7395 \end{bmatrix} ,$$

and this has rank 3. Therefore the system is observable: using θ measurements only we can get an estimate of both heave velocity w and pitch rate q (how to do this we will see later).

Now let's say we are interested in depth as well. The linear equation for the rate of change of submarine depth, z, is

$$\dot{z} = -U\theta + w .$$

If we incorporate this as our fourth state equation, the new A matrix is now 4×4 and B is 4×1. Keeping the same measurement, θ only, we have

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} .$$

If we compute the observabilty matrix $\mathcal{O}$, its rank is 3 instead of 4. Therefore, the system is unobsvervable and one state $(4 - 3 = 1)$ can not be estimated by looking at the angle θ only. This is, of course, z. If we assume that we have measurements of z only,

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} .$$

The new observability matrix has now full rank (4) which means that using a depth sensor only we should, in principle, be able to guess all the rest: θ, w, and q. The formalization of this "guess" constitutes the observer or estimator problem we discuss in Section 3.