CONTROL DESIGN BASED ON LYAPUNOV'S METHOD

There are basically two ways of using Lyapunov's method for control design. In the first way we assume a specific form of control law and then we find a Lyapunov function to justify or reject the choice. The third example of the notes on page 85 (i.e., the motion of a space vehicle about the principal axes of inertia) falls into this category. In the second way, we assume a Lyapunov function candidate for the system; i.e., one that is everywhere positive definite, and then try and come up with a control law to make this a real Lyapunov function; i.e., its time derivative is negative definite. We illustrate this with the following example:

Consider a second-order system with nonlinear spring and damper characteristics of the form,

$$\ddot{x} + b(\dot{x}) + c(x) = 0,$$

where b and c are continuous functions which satisfy the following two conditions,

$$\dot{x}b(\dot{x}) > 0 \text{ for } \dot{x} \neq 0, \text{ and}$$

$$xc(x) > 0 \text{ for } x \neq 0.$$

These equations simply state that physically realizable dampers and springs must produce forces which will tend to oppose their velocity or displacement. We can use Lyapunov's method to show that such systems are globally asymptotically stable. Note that together with the continuity assumptions, the sign conditions on the functions b and c imply that $b(0)=0$ and $c(0)=0$. A positive definite function for this system is,

$$V = \tfrac{1}{2}\dot{x}^2 + \int_0^x c(y)dy,$$

which is simply the sum of the potential and kinetic energy of the system. Differentiating V we obtain,

$$\dot{V} = \dot{x}\ddot{x} + c(x)\dot{x} = -\dot{x}b(\dot{x}) - \dot{x}c(x) + c(x)\dot{x} = -\dot{x}b(\dot{x}) < 0.$$

Now consider the problem of stabilizing the system,

$$\ddot{x} - \dot{x}^3 + x^2 = u,$$

in other words bring it to equilibrium at $x=0$. First, we can attempt to do this using linear state feedback. If we linearize the equation, we get

$$\ddot{x} = u,$$

where a linear feedback control law is

$$u = -k_1 x - k_2 \dot{x}.$$

We can do pole placement to find the gains; if we specify the natural frequency of this second order system to be equal to 1 and its damping ratio equal to 0.7 we get,

$$u = -x - 1.4\dot{x}.$$

To check the response, we can simulate the full system,

$$\ddot{x} - \dot{x}^3 + x^2 = -x - 1.4\dot{x}.$$

One way to do this is to construct the block diagram as shown in the figure and plot the response for different initial conditions. Do the following:

1. Build a block diagram similar to the one shown in the figure and simulate its response.

2. Does the linear control law stabilize the system for arbitrary initial conditions?
3. For what set(s) of initial conditions will the linear control fail to stabilize the system?

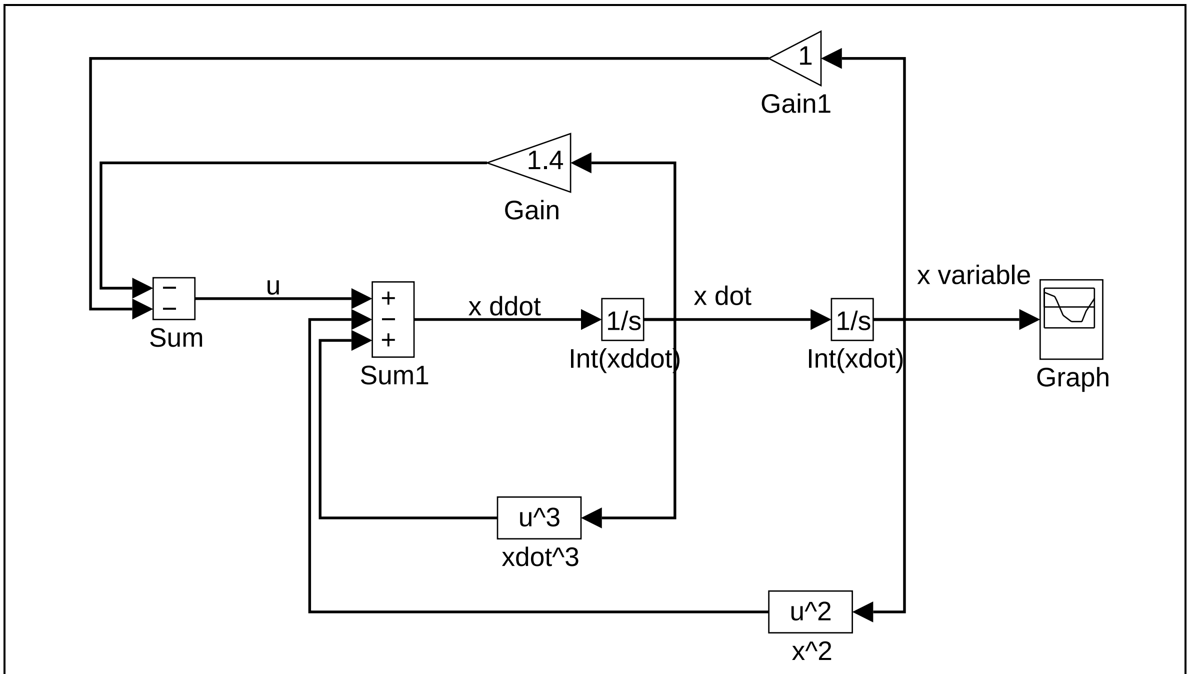

Figure: Block diagram implementation of the linear feedback law

Based on our previous discussion of the general nonlinear spring-mass-damper system, it is sufficient to choose a continuous nonlinear control law of the form,

$$u = u_1(x) + u_2(\dot{x}),$$

where u_1 and u_2 are nonlinear functions to be determined. Global asymptotic stability of the closed loop system is guaranteed if,

$$\dot{x}(\dot{x}^3 + u_2(\dot{x})) < 0 \text{ and } x(x^2 - u_1(x)) > 0.$$

Therefore, we can choose,

$$u_1 = x^2 - x \text{ and } u_2 = -\dot{x}^3 - 1.4\dot{x}.$$

Let's evaluate the response of this control law. Do the following:

4. Build a block diagram to simulate the nonlinear control law.
5. Simulate the response of the system for the same initial conditions as the linear control law.
6. Does this control law stabilize the system for all sets of initial conditions? Do you see a simple explanation for this?